

LOCAL CONTACT HOMOLOGY AND APPLICATIONS

UMBERTO L. HRYNIEWICZ AND LEONARDO MACARINI

ABSTRACT. We introduce a local version of contact homology for an isolated periodic orbit γ of the Reeb flow and prove that its rank is uniformly bounded for isolated iterations. Several applications are obtained, including a generalization of Gromoll-Meyer's theorem on the existence of infinitely many simple periodic orbits, resonance relations and conditions for the existence of non-hyperbolic periodic orbits.

1. INTRODUCTION

Since the seminal works of Floer [11, 12, 13] several Morse theoretic methods have been developed in order to understand periodic orbits in Hamiltonian dynamics. In particular, contact homology was introduced in [9] in the bigger framework of symplectic field theory. Its chain complex is generated by good periodic orbits of the Reeb flow graded by the reduced Conley-Zehnder index and the differential counts rigid pseudo-holomorphic curves in the symplectization asymptotic to closed orbits.¹ Recall that a periodic orbit is good if it is not an even multiple of a simple periodic orbit whose linearized first return map has an odd number of real eigenvalues less than minus one. Otherwise, it is called bad.

There are different versions of contact homology, see [2] for a survey. We will consider here both cylindrical contact homology and linearized contact homology. The first one is defined and an invariant of the contact structure for nice contact forms, that is, contact forms with no periodic orbits of degree one, zero or minus one. The latter one, in turn, does not impose any condition on the closed orbits, but requires an augmentation for a certain differential graded algebra, and depends on the homotopy class of the augmentation. Nice contact forms admit a trivial augmentation and the linearized contact homology with respect to this augmentation is isomorphic to cylindrical contact homology [2]. Hence cylindrical contact homology will be regarded here as a particular case of linearized contact homology. If the contact manifold admits a strong filling (X, ω) then it induces an augmentation over the rationals under the hypothesis that $\omega|_{\pi_2(X)} = 0$ and $c_1(TX, \omega) = 0$.

The purpose of this work is to introduce a local version of contact homology and provide some applications. For well known transversality reasons, our results are conditional on the completion of foundational work by Hofer, Wysocki and Zehnder, see [22, 23, 24].

1.1. Main result. Let us fix from now on a closed contact co-oriented manifold (N^{2n-1}, ξ) . Throughout this work we will *always assume that the first Chern class of the contact structure vanishes and consider only augmentations over the rationals*. Let α be a contact form for ξ and γ an isolated periodic orbit of the Reeb flow of α . This means that there is no sequence of periodic orbits converging to γ besides the obvious one.

Date: February 15, 2012.

¹One should be aware that for homotopically non-trivial periodic orbits the grading is not unambiguously defined and requires fixing an extra structure on (N, ξ) , see Section 2.1.2. Throughout this work we fix such structure.

One associate to γ its local contact homology $HC_*(\alpha, \gamma)$ by making a small non-degenerate perturbation of α and counting rigid holomorphic cylinders that are contained in the symplectization of an isolating neighborhood K of γ and are asymptotic to good periodic orbits homotopic to γ in K , see Sections 3 and 4 for details and properties of local contact homology. An interesting feature of local contact homology is that although we relate it to linearized contact homology via Morse inequalities (Proposition 7.4) it does not depend on the augmentation at all. This is due to the fact that there are no holomorphic planes with finite energy in the symplectization of K . Therefore, one cannot have holomorphic curves with more than one negative puncture.

The main result in this work establishes a uniform bound for the rank of local contact homology of iterations of a periodic orbit.

Theorem 1.1. *Let γ be a periodic orbit of the Reeb flow such that γ^j is isolated for every $j \in \mathbb{N}$. Then there exists a constant $B > 0$ satisfying $\dim HC_*(\alpha, \gamma^j) < B$ for every $j \in \mathbb{N}$.*

The proof is given in Section 6.6 and is based on two main building blocks. The first one is Proposition 6.1 establishing that the rank of the local contact homology of an isolated periodic orbit is less or equal than the rank of the local Floer homology of its return map. The second one is the main result in [16] that implies the existence of a uniform bound for the rank of local Floer homology of admissible iterations of an isolated periodic point of a Hamiltonian diffeomorphism.

1.2. Applications. Augmentations are defined for the differential graded algebra associated to a contact form, but we can define homotopy classes of augmentations for the contact *structure* as the following discussion shows. We refer to [2] for details. Given two non-degenerate contact forms α_0 and α_1 for ξ and regular almost complex structures J_0 and J_1 , a regular cobordism between the pairs (α_0, J_0) and (α_1, J_1) induces a chain map $\Psi : (\mathcal{A}(\alpha_0), \partial_0) \rightarrow (\mathcal{A}(\alpha_1), \partial_1)$ between the corresponding differential graded algebras. An augmentation ϵ for $(\mathcal{A}(\alpha_1), \partial_1)$ induces an augmentation $\Psi^*\epsilon$ for $(\mathcal{A}(\alpha_0), \partial_0)$ and Ψ induces an isomorphism $\tilde{\Psi}_\epsilon : HC^{[\Psi^*\epsilon]}(\alpha_0, J_0) \rightarrow HC^{[\epsilon]}(\alpha_1, J_1)$. It turns out that the homotopy class of $\Psi^*\epsilon$ does not depend on the choice of the cobordism (see [8, Theorem 3.2]) and hence we have a natural identification of homotopy classes of augmentations for non-degenerate contact forms for ξ . In this way, one can define the set of homotopy classes of augmentations for ξ as the set of equivalence classes under this identification. The collection of the corresponding linearized contact homologies is an invariant of the contact structure, see [2, Theorem 2.8]. We will abuse a bit the notation and denote such equivalence classes by $[\epsilon]$ and its corresponding linearized contact homology by $HC_*^{[\epsilon]}(\xi)$. Let $b_*^{[\epsilon]}(\xi)$ be the rank of $HC_*^{[\epsilon]}(\xi)$ (that may be infinite).

We say that ξ is homologically unbounded if there is a homotopy class of augmentations $[\epsilon]$ for ξ such that there is a sequence of integers $|l_i| \rightarrow \infty$ satisfying $\lim_{i \rightarrow \infty} b_{l_i}^{[\epsilon]}(\xi) = \infty$. The following theorem follows easily from Theorem 1.1 and the Morse inequalities in Proposition 7.4. Its proof is given in Section 8.1.

Theorem 1.2. *Suppose that ξ is homologically unbounded. Then the Reeb flow of every contact form α for ξ has infinitely many geometrically distinct periodic orbits.*

Examples of homologically unbounded contact structures can be obtained by cosphere bundles. More precisely, given a closed oriented manifold M of dimension n , it is proved in

[7] that

$$(1.1) \quad HC_{*+(n-3)}^{[\epsilon_0]}(S^*M, \xi_0) \simeq H_*(\Lambda M/S^1, M; \mathbb{Q}),$$

where ξ_0 is the standard contact structure of the unit cotangent bundle S^*M , ϵ_0 is given by the obvious filling of S^*M , ΛM is the free loop space on M and $M \subset \Lambda M$ indicates the subset of constant loops [2, Theorem 4.4].² A result due to Vigué-Poirrier and Sullivan [31] establishes that if M is simply connected then the rank of $H_*(\Lambda M/S^1, M; \mathbb{Q})$ is unbounded if and only if the homological algebra of M is not generated by a single class. Consequently, we have the following generalization of a celebrated result due to Gromoll and Meyer [18]. It is also proved in [28] for general coefficient fields using symplectic homology.

Corollary 1.3. *Let M be a closed oriented manifold such that the rank of $H_*(\Lambda M/S^1, M; \mathbb{Q})$ is unbounded. Then every fiberwise starshaped hypersurface in T^*M has infinitely many geometrically distinct periodic orbits. In particular, the result holds if M is simply connected and its homological algebra over \mathbb{Q} is not generated by a single class.*

Another source of examples is given by connected sums. Given two contact manifolds (N_1, ξ_1) and (N_2, ξ_2) it is well known that its connected sum $N_1 \# N_2$ carries a contact structure $\xi_1 \# \xi_2$, see [14]. Moreover, homotopy classes of augmentations $[\epsilon_1]$ and $[\epsilon_2]$ for ξ_1 and ξ_2 respectively induce a homotopy class of augmentations $[\epsilon_1 \# \epsilon_2]$ for $\xi_1 \# \xi_2$. A result due to Bourgeois and van Koert [2, 6] gives the long exact sequence

$$\begin{aligned} \cdots \rightarrow HC_{*-1}(S^{2n-3}, \xi_0) &\rightarrow HC_*^{[\epsilon_1 \# \epsilon_2]}(N_1 \# N_2, \xi_1 \# \xi_2) \\ &\rightarrow HC_*^{[\epsilon_1]}(N_1, \xi_1) \oplus HC_*^{[\epsilon_2]}(N_2, \xi_2) \rightarrow HC_{*-2}(S^{2n-3}, \xi_0) \rightarrow \cdots \end{aligned}$$

where $HC_*(S^{2n-3}, \xi_0)$ is the cylindrical contact homology of the standard contact structure on S^{2n-3} . Since the rank of $HC_*(S^{2n-3}, \xi_0)$ is at most one, we conclude that the connected sum of any contact manifold (admitting an augmentation) with a homologically unbounded contact manifold is homologically unbounded.

Now, fix a non-degenerate contact form α for ξ with an augmentation ϵ . There is a natural filtration in contact homology given by the action. Given $a \in \mathbb{R}$ denote the truncated homology by $HC_*^{a, \epsilon}(\alpha)$. Notice that it in general depends on the contact form and the augmentation. Following [30, 27], we define the growth rate of $HC_*^\epsilon(\alpha)$ as

$$\Gamma^\epsilon(\alpha) = \limsup_{a \rightarrow \infty} \frac{1}{\log a} \log \dim \iota(HC_*^{a, \epsilon}(\alpha)),$$

where $\iota : HC_*^{a, \epsilon}(\alpha) \rightarrow HC_*^{[\epsilon]}(\xi)$ is the map induced by the inclusion. The argument in [30, Section 4a] shows that the set $\{\Gamma^\epsilon(\alpha); \epsilon \text{ is an augmentation for } \alpha\}$ is an invariant of the contact structure. Since our context is different from the one in [30] (in particular, we have to deal with augmentations) we will give a proof of this fact in Section 8.2. The following theorem will be proved in Section 8.3.

Theorem 1.4. *If there exists a non-degenerate contact form α for ξ with an augmentation ϵ such that $\Gamma^\epsilon(\alpha) > 1$ then every contact form α representing ξ has infinitely many geometrically distinct periodic orbits.*

²The trivializations of the contact structure over periodic orbits used in [7] send the vertical distribution in the cotangent bundle to a fixed Lagrangian subspace in \mathbb{R}^{2n-2} . This fixes the grading in the isomorphism (1.1).

There are several examples of contact manifolds satisfying the previous hypothesis, see [25].

Our next application is a generalization of a result due to Ginzburg and Kerman [17] on resonance relations. Assume that there exist integers l_- and l_+ such that $HC_l^{[\epsilon]}(\xi)$ has finite rank for every $l \leq l_-$ and $l \geq l_+$. Under this assumption the positive/negative mean Euler characteristic is defined as

$$\chi_{\pm}^{[\epsilon]}(\xi) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{l=|l_{\pm}|}^m (-1)^l b_{\pm l}^{[\epsilon]}(\xi)$$

provided that the limits exist. Notice that by Theorem 1.2 if ξ admits a contact form with finitely many prime periodic orbits then this hypothesis is fulfilled and the limits above always exist.

Given an isolated periodic orbit γ , its positive/negative local Euler characteristic is defined as

$$\chi_{\pm}(\alpha, \gamma) = \sum_{\pm i \geq 0} (-1)^i \dim HC_i(\alpha, \gamma).$$

The sum above is finite. Now, assume that γ^j is isolated for every $j \in \mathbb{N}$. The positive/negative local *mean* Euler characteristic of γ is defined as

$$\hat{\chi}_{\pm}(\alpha, \gamma) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=1}^m \chi_{\pm}(\alpha, \gamma^j).$$

By Theorem 1.1 these limits exist.

Theorem 1.5. *Let α be a contact form for ξ with finitely many simple closed orbits. Given any homotopy class of augmentations $[\epsilon]$ for ξ , the positive/negative mean Euler characteristic satisfies*

$$\chi_{\pm}^{[\epsilon]}(\xi) = \sum_{\pm \Delta(\gamma) > 0} \frac{\hat{\chi}_{\pm}(\alpha, \gamma)}{\Delta(\gamma)},$$

where $\Delta(\gamma)$ is the mean index of γ and the sum runs over the set of simple periodic orbits γ such that $\pm \Delta(\gamma) > 0$.

Notice that in the previous theorem we do not assume α to be non-degenerate. When α is non-degenerate, the local mean Euler characteristic of a periodic orbit is easily computed and we obtain

Corollary 1.6 (Theorem 1.7 and Remark 1.10 in [17]). *If α is non-degenerate and has finitely many simple periodic orbits then*

$$\chi_{\pm}^{[\epsilon]}(\xi) = \frac{1}{2} \sum_{\gamma \in B^{\pm}(\alpha)} \frac{(-1)^{|\gamma|}}{\Delta(\gamma)} + \sum_{\gamma \in G^{\pm}(\alpha)} \frac{(-1)^{|\gamma|}}{\Delta(\gamma)},$$

where $B^{\pm}(\alpha)$ (resp. $G^{\pm}(\alpha)$) is the set of simple periodic orbits with positive/negative mean index whose even iterates are bad (resp. good).

An application of the previous theorem is the following result. Recall that a periodic orbit is hyperbolic if its linearized first return map has no eigenvalue in the circle.

Theorem 1.7. *Suppose that there is a homotopy class of augmentations $[\epsilon]$ for ξ such that $HC_{n-3}^{[\epsilon]}(\xi)$ has finite rank and that there exists a positive integer C such that*

$$(-1)^n \sum_{i=0}^{mC} (-1)^{\pm i + n - 3} b_{\pm i + n - 3}^{[\epsilon]}(\xi) < (-1)^n mC \chi_{\pm}^{[\epsilon]}(\xi)$$

for every $m \in \mathbb{N}$. If a contact form α for ξ has finitely many geometrically distinct closed orbits then there is a non-hyperbolic one.

Examples satisfying these hypotheses can be obtained using Yau's computation of the contact homology of subcritical Stein fillable contact manifolds [34]. More precisely, it is proved in [34] that given a subcritical Stein domain (V^{2n}, J) such that $\partial V = N$ then the cylindrical contact homology is given by

$$HC_*(\xi) \simeq \oplus_{m \in \mathbb{N}^0} H_{2(n+m-1)-*}(V),$$

where ξ is the maximal complex subbundle of TN . One can check from this that if n is even and V has trivial homology in every odd degree then N satisfies the hypotheses of Theorem 1.7 for the positive Euler characteristic. In particular, we get S^{2n-1} with its standard contact structure and n even, recovering a result obtained by Viterbo in [32].

As a byproduct of the proof of Theorem 1.7 we obtain

Theorem 1.8. *Suppose that there is a homotopy class of augmentations $[\epsilon]$ for ξ such that*

$$0 < \dim HC_{n-3}^{[\epsilon]}(\xi) < \infty.$$

If a contact form α for ξ has finitely many geometrically distinct closed orbits then there is a non-hyperbolic one.

A homology computation in [1] shows that there is a family of inequivalent contact structures on $S^2 \times S^3$ meeting this assumption. The isomorphism (1.1) implies that the unit cotangent bundle of a closed oriented manifold with non-trivial fundamental group and compact universal covering also satisfies this condition. By the aforementioned long exact sequence, the connected sum of one of these manifolds with any contact manifold (N, ξ) such that $\dim HC_{n-3}^{[\epsilon]}(\xi) < \infty$ has a contact structure satisfying this assumption as well.

Organization of the paper. Section 2 furnishes the basic material on pseudo-holomorphic curves necessary for this work. In Section 3 we define isolating neighborhoods of isolated closed Reeb orbits. This notion is crucial in the construction of local contact homology accomplished in Section 4. The computation of local contact homology is focused in Sections 5 and 6 where we deal with prime and iterated periodic orbits respectively. Morse inequalities are achieved in Section 7. They are a cornerstone in the proof of our applications presented in Section 8. Finally, Appendix A provides technical details about holomorphic curves in symplectizations of stable Hamiltonian structures.

Note. After we started to write the present paper, we were aware that Mark McLean obtained similar results using symplectic homology [28]. The results can be related to ours using the Bourgeois-Oancea long exact sequence [2]. However, although our techniques lead to similar results, we think they are complementary to McLean's since they furnish an equivariant version of the local homology of closed Reeb orbits.

Acknowledgments. We are grateful to Alberto Abbondandolo, Urs Frauenfelder, Viktor Ginzburg, Nancy Hingston, Alexandru Oancea, Gabriel Paternain and Otto van Koert for

useful discussions regarding this paper. Special thanks to Alberto Abbondandolo for pointing out to us, in the beginning of this work, Ginzburg-Gurel's result in [16] and its relationship with Gromoll-Meyer's theorem and to Viktor Ginzburg for pointing out a mistake in a first draft of this paper. We also thank the IAS for its hospitality during part of the preparation of this work.

These results were first presented by the first author at the Workshop on Conservative Dynamics and Symplectic Geometry, IMPA, Rio de Janeiro, Brazil, August 1–5, 2011. He thanks the organizers for the opportunity to participate in such a great event.

This material is based upon work supported by the National Science Foundation under agreement No. DMS-0635607. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

2. PRELIMINARIES FOR LOCAL CONTACT HOMOLOGY

2.1. Stable Hamiltonian structures. We start by recalling the concept of a stable Hamiltonian structure from [3].

Definition 2.1. *A stable Hamiltonian structure on a $(2n - 1)$ -manifold N is a triple $\mathcal{H} = (\xi, X, \omega)$ where $\xi \subset TN$ is a hyperplane distribution, X is a vector field everywhere transverse to ξ and its flow preserves ξ , ω is a closed 2-form such that $\omega|_\xi$ turns ξ into a symplectic vector bundle and $i_X\omega = 0$.*

We refer to X as the Hamiltonian vector field of \mathcal{H} .

Remark 2.2. *Given such \mathcal{H} one can define a 1-form λ on N by*

$$(2.1) \quad i_X\lambda = 1, \quad \ker \lambda = \xi.$$

It is easy to check that $\lambda \wedge \omega^n$ is a volume form and $\ker d\lambda \supseteq \ker \omega = \mathbb{R}X$. In particular $\mathcal{L}_X\lambda = 0$ and $\mathcal{L}_X\omega = 0$. The stable Hamiltonian structure could be alternatively defined as the pair (λ, ω) and, in this case, X and ξ would be uniquely determined by $i_X\lambda = 1$, $i_X\omega = 0$ and $\xi = \ker \lambda$.

Any contact form α on N induces a stable Hamiltonian structure $(\xi, R, C d\alpha)$, where R is the associated Reeb vector field, $\xi = \ker \alpha$ is the contact structure and $C > 0$.

Throughout Section 2 we fix a compact $(2n - 1)$ -manifold N equipped with a stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega)$, and assume that $c_1(\xi, \omega)$ vanishes.

2.1.1. Periodic orbits. If $x : \mathbb{R} \rightarrow N$ is a periodic trajectory of X with period $T > 0$ then $x_T : S^1 \simeq \mathbb{R}/\mathbb{Z} \rightarrow N$, $t \mapsto x(Tt)$, defines an element of $C^\infty(S^1, N)$. A periodic orbit γ of X is the element of $C^\infty(S^1, N)/S^1$ induced by some x_T as above. We write $\gamma = (x, T)$ and $\gamma^k = (x, kT)$ for any $k \in \mathbb{Z}^+$. The set $x(\mathbb{R}) \subset N$ is called the geometric image of γ . If T is the minimal positive period of x then we call γ simply covered. The set of periodic orbits of X will be denoted by $\mathcal{P}(\mathcal{H})$. When \mathcal{H} is induced by some contact form α we may write $\mathcal{P}(\alpha)$, or simply \mathcal{P} when the context is clear. For any given $K \subset N$ we denote by $\mathcal{P}(\mathcal{H}, K)$, or $\mathcal{P}(\alpha, K)$, the subset of orbits with geometric image contained in K .

The flow $\{\phi_t\}_{t \in \mathbb{R}}$ of X induces a ω -symplectic linear flow on ξ

$$(2.2) \quad d\phi_t : \xi \rightarrow \xi.$$

If $\gamma = (x, T) \in \mathcal{P}$ and 1 is not in the spectrum of $d\phi_T : \xi_{x(0)} \rightarrow \xi_{x(0)=x(T)}$ then γ is called non-degenerate. When every $\gamma \in \mathcal{P}(\mathcal{H})$ (or $\gamma \in \mathcal{P}(\mathcal{H}, K)$) is non-degenerate we call \mathcal{H}

non-degenerate (on K). The notation $\gamma = (x, T)$ is ambiguous since the choice of x is not determined: we choose a special point pt_γ in the geometric image of every closed orbit γ of $\{\phi_t\}$ and assume $x(0) = \text{pt}_\gamma$. Orbits with the same geometric image share the same special point.

2.1.2. Conley-Zehnder indices. Assume for simplicity that $H_1(N, \mathbb{Z})$ is torsion free, and choose a set of generators $\{C_i\}$, $i = 1, \dots, l$. We assume the C_i are represented by 1-dimensional submanifolds, still denoted C_i , and we choose ω -symplectic trivializations of $\xi|_{C_i}$. These choices will be fixed for the rest of this work.

Any $\gamma = (x, T) \in \mathcal{P}(\mathcal{H})$ can be seen as a singular 1-chain, which induces a homology class $[\gamma] \in H_1(N, \mathbb{Z})$. There are unique $n_i \in \mathbb{Z}$ satisfying $[\gamma] = \sum_i n_i C_i$. A 2-chain realizing a homology between γ and $\sum_i n_i C_i$ can be used to single out a homotopy class of ω -symplectic trivializations of $(x_T)^*\xi$. A trivialization in this class represents the linearized dynamics of X as a path in the group $Sp(2n-2)$ starting at the identity. If γ is non-degenerate, this path ends in the complement of the Maslov cycle, and has a well-defined Conley-Zehnder index as defined in [29], denoted $\mu_{\text{CZ}}(\gamma) \in \mathbb{Z}$. This is independent of the choice of the 2-chain since we assume $c_1(\xi)$ vanishes.

It is convenient to consider the degree of γ defined by

$$(2.3) \quad |\gamma| := \mu_{\text{CZ}}(\gamma) + n - 3.$$

2.1.3. Good orbits. Let $\gamma = (x, T) \in \mathcal{P}(\mathcal{H})$ be simply covered. According to [9], if the number of eigenvalues of $d\phi_T : \xi_{x(0)} \rightarrow \xi_{x(T)=x(0)}$ in $(-1, 0)$ is odd (counted with multiplicities) then the even multiples γ^{2k} are called bad orbits. An orbit is called good if it is not bad, and we define

$$(2.4) \quad \mathcal{P}_0(\mathcal{H}) := \{\gamma \in \mathcal{P}(\mathcal{H}) : \gamma \text{ is good}\}$$

and $\mathcal{P}_0(K, \mathcal{H}) = \mathcal{P}(K, \mathcal{H}) \cap \mathcal{P}_0(\mathcal{H})$. In the case \mathcal{H} is induced by a contact form α we write $\mathcal{P}_0(\alpha)$ and $\mathcal{P}_0(K, \alpha)$ accordingly.

2.2. Pseudo-holomorphic curves. We take a moment to review the basic definitions from pseudo-holomorphic curve theory.

2.2.1. Cylindrical almost complex structures. Let V be a compact $(2n-1)$ -manifold. In $\mathbb{R} \times V$ there is a natural \mathbb{R} -action induced by the maps

$$(2.5) \quad \tau_c : (a, p) \mapsto (a + c, p), \quad c \in \mathbb{R}.$$

In the language of [3], an almost complex structure J on $\mathbb{R} \times V$ is cylindrical if $\tau_c^* J = J$, $\forall c \in \mathbb{R}$, and if the vector field $R := J\partial_a$ is horizontal, i.e. it is tangent to $\{a\} \times V$, $\forall a \in \mathbb{R}$. Since J is \mathbb{R} -invariant, the formula $\Xi := TV \cap J(TV)$ defines a $(2n-2)$ -dimensional J -invariant distribution in V , and R seen as vector field in V is everywhere transverse to Ξ . Note also that J is a complex structure on Ξ . J is called symmetric if the 1-form λ on V defined by $i_R \lambda = 1$ and $\Xi = \ker \lambda$ satisfies $\mathcal{L}_R \lambda = i_R d\lambda = 0$. Let Ω be a closed 2-form on V of maximal rank, that is, $\dim \ker \Omega = 1$. Then a cylindrical J as above is said to be adjusted to Ω if the restriction $\Omega|_\Xi$ turns Ξ into a symplectic bundle, $J|_\Xi$ is $\Omega|_\Xi$ -compatible, that is $\Omega(\cdot, J\cdot)$ defines a (fiberwise) metric on Ξ , and $i_R \Omega = 0$. Note that if J is cylindrical, symmetric and adjusted to Ω as above then $\mathcal{H} = (\Xi, R, \Omega)$ is a stable Hamiltonian structure.

Conversely, a stable Hamiltonian structure $\mathcal{H} = (\Xi, R, \Omega)$ induces symmetric cylindrical almost complex structures on $\mathbb{R} \times N$ adjusted to Ω . In fact, choose some Ω -compatible complex structure \hat{J} on Ξ . Then we have a unique \mathbb{R} -invariant almost complex structure J

on $\mathbb{R} \times N$ defined by requiring $J\partial_a = R$ and $J|_{\Xi} = \hat{J}$, which is the desired almost complex structure. The set of such J will be denoted by $\mathcal{J}(\mathcal{H})$. When \mathcal{H} is induced by a contact form α we may write $\mathcal{J}(\alpha)$ instead of $\mathcal{J}(\mathcal{H})$.

2.2.2. Almost complex structures in non-cylindrical cobordisms. Consider stable Hamiltonian structures \mathcal{H}^\pm on N , and $J^\pm \in \mathcal{J}(\mathcal{H}^\pm)$. For a given $L > 0$ we denote by $\mathcal{J}_L(J^-, J^+)$ the space of almost complex structures \bar{J} on $\mathbb{R} \times N$ such that $\bar{J}|_{[L, +\infty)} \equiv J^+$ and $\bar{J}|_{(-\infty, -L]} \equiv J^-$. We set $\mathcal{J}(J^-, J^+) = \cup_{L>0} \mathcal{J}_L(J^-, J^+)$. When \mathcal{H}^\pm , $J^\pm \in \mathcal{J}(\mathcal{H}^\pm)$ and $L > 0$ are fixed we may consider smooth 1-parameter families $\{\bar{J}^\tau\}_{\tau \in [0,1]} \subset \mathcal{J}_L(J^-, J^+)$. We denote the spaces of such families by $\mathcal{J}_{\tau,L}(J^-, J^+)$ and $\mathcal{J}_\tau(J^-, J^+) = \cup_{L>0} \mathcal{J}_{\tau,L}(J^-, J^+)$.

We need to consider almost complex structures in splitting cobordisms. Fix \mathcal{H} , $J \in \mathcal{J}(\mathcal{H})$ and numbers $0 < L < R$. We denote by $\mathcal{J}_{L<R}(J)$ the set of almost complex structures which coincide with J on $(\mathbb{R} \setminus [-L-R, L+R]) \times N$, and are cylindrical on the neck $[L-R, R-L] \times N$. Clearly $\mathcal{J}_{L<R}(J) \subset \mathcal{J}_{L+R}(J, J)$. Also, we consider the set $\mathcal{J}_{\tau,L<R}(J)$ of smooth families $\{\bar{J}_t\}_{t \in [0,1]} \subset \mathcal{J}_{L<R}(J)$. Note that we do not assume elements in $\mathcal{J}_{L<R}(J)$ to be symmetric on the neck $[L-R, R-L] \times N$.

2.2.3. Finite-energy curves. Let (S, j) be a closed Riemann surface and $Z \subset S$ be a finite set. Fix $J \in \mathcal{J}(\mathcal{H})$. A smooth map $F = (a, f) : S \setminus Z \rightarrow \mathbb{R} \times N$ is J -holomorphic (or pseudo-holomorphic) if it satisfies the Cauchy-Riemann equation

$$\bar{\partial}_J(F) = \frac{1}{2}(dF + J \cdot dF \cdot j) = 0.$$

Consider the set $\Lambda = \{\phi : \mathbb{R} \rightarrow [0, 1] \mid \phi' \geq 0\}$. We can view an element of Λ as a real function on $\mathbb{R} \times N$ that depends only on the first coordinate. Similarly we can view the form λ in (2.1) as a 1-form on $\mathbb{R} \times N$. Following [3], one defines the ω -energy of F as

$$(2.6) \quad E_\omega(F) = \int_{S \setminus Z} f^* \omega$$

and the energy of F as

$$(2.7) \quad E(F) = E_\omega(F) + \sup_{\phi \in \Lambda} \int_{S \setminus Z} F^*(d\phi \wedge \lambda).$$

All these integrals have non-negative integrands. If $0 < E(F) < \infty$ then F is said to be a finite-energy pseudo-holomorphic curve. The elements of Z are called punctures of F , and a puncture $z \in Z$ is called removable when F is bounded near z . In this case, an application of Gromov's Removable Singularity Theorem shows that F can be smoothly continued across z .

Let $\mathcal{H}^\pm = (\xi^\pm, X^\pm, \omega^\pm)$ be stable Hamiltonian structures on N , fix $J^\pm \in \mathcal{J}(\mathcal{H}^\pm)$, $L > 0$ and $\bar{J} \in \mathcal{J}_L(J^-, J^+)$. Assume also there exists a symplectic form Ω on $[-L, L] \times N$ that agrees with ω^\pm on $T(\{\pm L\} \times N)$, up to positive constants, and tames \bar{J} . Consider a smooth map $F : S \setminus Z \rightarrow \mathbb{R} \times N$ which is \bar{J} -holomorphic. Following [3] we define

$$(2.8) \quad \begin{aligned} E(F) = & \int_{F^{-1}([-L, L] \times N)} F^* \Omega + \sup_{\phi \in \Lambda} \int_{F^{-1}([L, +\infty) \times N)} F^*(d\phi \wedge \lambda^+ + \omega^+) \\ & + \sup_{\phi \in \Lambda} \int_{F^{-1}((-\infty, -L] \times N)} F^*(d\phi \wedge \lambda^- + \omega^-) \end{aligned}$$

where λ^\pm are the 1-forms associated to \mathcal{H}^\pm as in (2.1). All integrands above are non-negative. Moreover F is constant if, and only if, $E(F) = 0$. F is called a finite-energy curve when

$0 < E(F) < \infty$. Again the points of Z are called punctures, and a puncture is removable if, and only if, F is bounded around it.

Finally, we need to define finite-energy \tilde{J} -holomorphic curves for $\tilde{J} \in \mathcal{J}_{L < R}(J)$, where $0 < L < R$, $\mathcal{H} = (\xi, X, \omega)$ and $J \in \mathcal{J}(\mathcal{H})$ are fixed. The correct taming conditions are as follows. We assume \tilde{J} is adjusted to some 2-form $\bar{\omega} \in \Omega^2(N)$ of maximal rank on the neck³ $[L - R, R - L] \times N$, as discussed in 2.2.1, and assume also that \tilde{J} is tamed by symplectic forms Ω^\pm on $(\pm[R + L, R - L]) \times N$ satisfying the following conditions:

- Ω^+ coincides with ω on $T(\{R + L\} \times N)$ and with $\bar{\omega}$ on $T(\{R - L\} \times N)$ up to positive constants,
- Ω^- coincides with ω on $T(\{-R - L\} \times N)$ and with $\bar{\omega}$ on $T(\{-R + L\} \times N)$ up to positive constants.

Any $\tilde{J} \in \mathcal{J}_{L < R}(J)$ induces a 1-form $\bar{\lambda}$ on N by $i_{\bar{R}}\bar{\lambda} = 1$, $\bar{\xi} = \ker \bar{\lambda}$, where $\bar{\xi}$ is the maximal complex subbundle of TN induced by the cylindrical piece $\bar{J}|_{[L-R, R-L] \times N}$ and $\bar{R} = \bar{J}|_{[L-R, R-L] \times N} \cdot \partial_t$ (here t is the \mathbb{R} -coordinate). Note that $(\bar{\xi}, \bar{R}, \bar{\omega})$ is **not** a stable Hamiltonian structure since \bar{J} does not need to be symmetric on the neck $[L - R, R - L] \times N$. The energy $E(F)$ of the \tilde{J} -holomorphic map $F = (a, f) : S \setminus Z \rightarrow \mathbb{R} \times N$ is

$$\begin{aligned}
 E(F) = & \sup_{\phi \in \Lambda} \int_{F^{-1}([L+R, +\infty) \times N)} F^*(d\phi \wedge \lambda + \omega) \\
 & + \sup_{\phi \in \Lambda} \int_{F^{-1}((-\infty, -L-R] \times N)} F^*(d\phi \wedge \lambda + \omega) \\
 (2.9) \quad & + \sup_{\phi \in \Lambda} \int_{F^{-1}([L-R, R-L] \times N)} F^*(d\phi \wedge \bar{\lambda} + \bar{\omega}) \\
 & + \int_{F^{-1}([-R-L, L-R] \times N)} F^*\Omega^- \\
 & + \int_{F^{-1}([R-L, R+L] \times N)} F^*\Omega^+.
 \end{aligned}$$

The behavior near the punctures is exactly as in the other cases.

2.2.4. Asymptotic behavior. Let $F : (S \setminus Z, j) \rightarrow \mathbb{R} \times N$ be a finite-energy J -holomorphic curve, where $J \in \mathcal{J}(\mathcal{H})$. The behavior of F near a non-removable puncture $z \in Z$ is studied in [21], see also the Appendix of [3]. Let $\psi : (B_1(0), 0) \rightarrow (V, z)$ be a holomorphic chart of (S, j) and write $F(s, t) = (a(s, t), f(s, t)) = F \circ \psi(e^{-2\pi(s+it)})$ for $(s, t) \in \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}$.

Proposition 2.3. *When X is non-degenerate one finds $\gamma = (x, T) \in \mathcal{P}(\mathcal{H})$, $\epsilon = \pm 1$, $c, d \in \mathbb{R}$ such that the loops $t \mapsto f(s, t)$ converge to $t \mapsto x(\epsilon T t + c)$ in $C^\infty(S^1, N)$, and all partial derivatives of $a(s, t) - \epsilon T s - d$ tend to 0 uniformly in t as $s \rightarrow +\infty$.*

A non-removable puncture z is positive if $\epsilon = +1$, and negative if $\epsilon = -1$. In any case one says that F is asymptotic to γ at z , and γ is the asymptotic limit of F at z . These definitions are independent of the choice of ψ .

Under the assumption that the Hamiltonian vector fields are non-degenerate, the asymptotic behavior of finite-energy curves in non-cylindrical cobordisms is analogous and we will not describe it here. The reader can easily guess the precise statements in this case.

³Note here that \tilde{J} is cylindrical on the neck.

3. ISOLATING NEIGHBORHOODS OF ISOLATED ORBITS

In this section we discuss the necessary geometric set-up for defining local contact homology of an isolated orbit. Let $\mathcal{H} = (\xi, X, \omega)$ be a stable Hamiltonian structure on the $(2n-1)$ -manifold N , and $x : \mathbb{R} \rightarrow N$ be a T -periodic trajectory of the vector field X , $T > 0$. Assume $\gamma = (x, T)$ is isolated, that is, $t \mapsto x(Tt)$ defines an isolated point in $C^\infty(S^1, N)/S^1$.

Definition 3.1. *An isolating neighborhood for γ is a compact connected neighborhood K of $x_0(\mathbb{R})$ with smooth boundary satisfying:*

- γ is the only closed X -orbit in K in the free homotopy class of γ (of loops in K),
- $H^2(K, \mathbb{R}) = 0$, $H^1(K, \mathbb{R}) = \mathbb{R}$ and there is a non-vanishing class $[\theta] \in H^1(K, \mathbb{R})$ represented by a closed 1-form θ satisfying $\inf_K i_X \theta > 0$.

There are no closed X -orbits inside K as above, which are contractible in K .

Lemma 3.2. *Every isolated γ has an isolating neighborhood.*

Proof. Let T_0 be the minimal positive period of x . Take a neighborhood $K \simeq S^1 \times \overline{B}$ of $x(\mathbb{R})$, where $B \subset \mathbb{R}^{2n-2}$ is an open ball around the origin, equipped with coordinates (t, z) , such that $x(t) \simeq (t/T_0, 0)$ and $\xi_{x(t)} \simeq 0 \times \mathbb{R}^{2n-2}$. Thus $\lambda|_{S^1 \times 0} = T_0 dt$. After shrinking B we get $X \pitchfork \{t\} \times \overline{B} \forall t$, so dt is a closed 1-form generating $H^1(K, \mathbb{R}) \simeq \mathbb{R}$ such that $i_X dt > 0$. Suppose $\gamma_k = (x_k, T_k) \neq \gamma$ are closed X -orbits homotopic to γ in K such that $\limsup_{k \rightarrow \infty} \sup_{t \in \mathbb{R}} |\pi_{\mathbb{R}^{2n-2}} \circ x_k(t)| = 0$, where $\pi_{\mathbb{R}^{2n-2}}$ denotes projection onto the second coordinate. We can assume $x_k(0) \in \{0\} \times \overline{B}$, so that $x_k \rightarrow x$ in C_{loc}^∞ . If $T_k \rightarrow \infty$ then $\int_{\gamma_k} dt = \int_{\gamma_k} dt \rightarrow +\infty$, an absurd. Thus we get a bound for the periods $T_k = \int_{\gamma_k} \lambda$ so that $\gamma_k \rightarrow \gamma$, contradicting the hypothesis that γ is isolated. This contradiction shows that, possibly after further shrinking \overline{B} , we get an isolating neighborhood K as in Definition 3.1 since $H^1(K, \mathbb{R}) = 0$. \square

Lemma 3.3. *Let K be an isolating neighborhood for the isolated orbit γ . For every neighborhood \mathcal{V} of γ in $C^\infty(S^1, N)/S^1$ there exists a neighborhood \mathcal{O} of \mathcal{H} in the space of stable Hamiltonian structures such that the following holds: if $\mathcal{H}' = (\xi', X', \omega') \in \mathcal{O}$ then all closed X' -orbits in K homotopic to γ (through loops in K) lie in \mathcal{V} .*

The statement above is easy and left with no proof.

3.1. Finite-energy cylinders in isolating neighborhoods.

Lemma 3.4. *Let $\mathcal{H} = (\xi, X, \omega)$, $J \in \mathcal{J}(\mathcal{H})$, $L > 0$ and an isolated X -orbit γ be given. Let also K be an isolating neighborhood of γ and Ω be a symplectic form on $[-L, L] \times K$ which tames J and agrees with ω on $T(\{\pm L\} \times K)$ up to positive constant. Then there are C^∞ -neighborhoods \mathcal{O} of \mathcal{H} and \mathcal{D} of Ω , and a neighborhood \mathcal{U} of J in the C^∞ -strong topology such that the following holds. If $\mathcal{H}^\pm = (\xi^\pm, X^\pm, \omega^\pm) \in \mathcal{O}$, $J^\pm \in \mathcal{J}(\mathcal{H}^\pm) \cap \mathcal{U}$, $\tilde{J} \in \mathcal{J}_L(J^-, J^+) \cap \mathcal{U}$, and $\tilde{\Omega} \in \mathcal{D}$ is symplectic and coincides with ω^\pm on $T(\{\pm L\} \times K)$ up to positive constants, then the following assertions hold.*

- (1) $\tilde{\Omega}$ tames \tilde{J} on $[-L, L] \times K$.
- (2) There exists $C > 0$ such that every finite-energy \tilde{J} -holomorphic map $F = (a, f) : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times N$ satisfying
 - i) $f(\mathbb{R} \times S^1) \subset K$ and the loops $t \mapsto f(s, t)$ are homotopic to γ in K ;
 - ii) $\{+\infty\} \times S^1$ is a positive puncture and $\{-\infty\} \times S^1$ is a negative puncture of F ;

must also satisfy $E(F) \leq C$ and $\overline{f(\mathbb{R} \times S^1)} \subset \text{int}(K)$.

Here we use $\tilde{\Omega}$ to define $E(F)$ according to the discussion in 2.2.3

The proof below makes use of the results from Appendix A.

Proof. Suppose the lemma is not true. Then we find $\mathcal{H}_n^\pm = (\xi_n^\pm, X_n^\pm, \omega_n^\pm) \rightarrow \mathcal{H}$ in C^∞ , $J_n^\pm \in \mathcal{J}(\mathcal{H}_n^\pm)$, $\tilde{J}_n \in \mathcal{J}_L(J_n^-, J_n^+)$ such that $J_n^+, J_n^-, \tilde{J}_n \rightarrow J$ in the strong C^∞ -topology, Ω_n symplectic forms on $[-L, L] \times K$ coinciding with ω_n^\pm on $T(\{\pm L\} \times K)$ up to positive constants such that $\Omega_n \rightarrow \Omega$, and non-constant finite-energy \tilde{J}_n -holomorphic maps $F_n = (a_n, f_n)$ satisfying i), ii) above, and not satisfying the conclusions of lemma.

Obviously, Ω_n tames \tilde{J}_n on $[-L, L] \times K$ if n is large enough. We claim that $\{E(F_n)\}$ is bounded. In fact, since $H_2(K, \mathbb{R})$ vanishes, there exists a primitive α for Ω on $[-L, L] \times K$. Using the Mayer-Vietoris principle, we find a sequence of primitives α_n , $d\alpha_n = \Omega_n$, on $[-L, L] \times K$ such that $\alpha_n \rightarrow \alpha$ in C^∞ . Consider the inclusions $i_\pm : K \simeq \{\pm L\} \times K \hookrightarrow \mathbb{R} \times K$, and the 1-forms $\alpha_n^\pm = i_\pm^* \alpha_n$, $\alpha^\pm = i_\pm^* \alpha$. Then, by the properties of Ω_n and Ω we have $\alpha_n^\pm \rightarrow \alpha^\pm$, $d\alpha_n^\pm = c_n^\pm \omega_n^\pm$, $d\alpha^\pm = c^\pm \omega$, where $c_n^\pm, c^\pm > 0$ satisfy $c_n^\pm \rightarrow c^\pm$. By adding $A\theta$ to α_n and α , with $A \gg 1$, we may assume $\liminf_n \inf_K i_{X_n^\pm} \alpha_n^\pm > 0$, where θ is a closed 1-form as in Definition 3.1. Abbreviating $U_n = F_n^{-1}([-L, L] \times K)$, and orienting $\partial^\pm U_n = F_n^{-1}(\{\pm L\} \times K)$ as the boundary of U_n , we get

$$\begin{aligned} & \int_{F_n^{-1}([L, +\infty) \times K)} f_n^* \omega_n^+ + \int_{U_n} F_n^* \Omega_n + \int_{F_n^{-1}((-\infty, -L] \times K)} f_n^* \omega_n^- \\ & \leq c' \left(\int_{F_n^{-1}([L, +\infty) \times K)} c_n^+ f_n^* \omega_n^+ + \int_{U_n} F_n^* \Omega_n + \int_{F_n^{-1}((-\infty, -L] \times K)} c_n^- f_n^* \omega_n^- \right) \\ & = c' \left(\int_{\gamma_n^+} c_n^+ \alpha_n^+ - \int_{\gamma_n^-} c_n^- \alpha_n^- \right) \leq c'' \end{aligned}$$

for some bounded constant $c'' > 0$ independent of n , where γ_n^\pm are the asymptotic limits of F_n at $\{\pm\infty\} \times S^1$ oriented by the Hamiltonian vector fields. By Lemma 3.3, $\gamma_n^\pm \rightarrow \gamma$, so that we get the uniform bound for the last line. The other terms of the energy defined in 2.2.3 are easier to estimate.

We can assume $\overline{f_n(\mathbb{R} \times S^1)} \cap \partial K \neq \emptyset$. By Lemma 3.3, $\gamma_n^\pm \subset \text{int}(K)$. Thus, we find (s_n, t_n) such that $f_n(s_n, t_n) \in \partial K$. We claim that dF_n is C_{loc}^0 -bounded. If not we use Lemma A.1 to get a non-constant finite-energy J -holomorphic plane with image in $\mathbb{R} \times K$. Lemma A.6 applied to the end of this plane gives us a contractible periodic orbit of X inside K , a contradiction to our assumptions.

Let $d_n = a_n(s_n, t_n)$ and define $\tilde{u}_n = (a_n(s + s_n, t_n) - d_n, f_n(s + s_n, t_n))$. By the above discussion we have C_{loc}^1 -bounds for \tilde{u}_n and, consequently, also C_{loc}^∞ -bounds. Up to a subsequence, we can assume there exists a finite-energy non-constant J -holomorphic cylinder $\tilde{u} = (b, u)$ such that $\tilde{u}_n \rightarrow \tilde{u}$ in C_{loc}^∞ . Letting $C \subset \mathbb{R} \times S^1$ be compact and arbitrary, we wish to show that $\int_C u^* \omega = 0$. By Lemma A.6 we find sequences $s_j^\pm \rightarrow \pm\infty$ such that the loops $u(s_j^\pm, t)$ converge to periodic orbits of X (homotopic to γ in K) as $j \rightarrow \infty$. But the only such orbit is γ itself, so that

$$0 \leq \int_C u^* \omega \leq \lim_j \int_{[s_j^-, s_j^+] \times S^1} u^* \omega = \frac{1}{c^\pm} \left(\int_\gamma \alpha^+ - \int_\gamma \alpha^+ \right) = 0.$$

Thus $E_\omega(\tilde{u}) = 0$ and we can apply Lemma A.5 to \tilde{u} to conclude that X has a periodic orbit in K , homotopic to γ in K , which touches ∂K . This is absurd. \square

We need a version of the above statement for almost complex structures as we stretch the neck. The proof is similar and omitted.

Lemma 3.5. *Suppose we have a stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega)$ on N , and let γ be an isolated closed X -orbit. Consider an isolating neighborhood K for γ , $J \in \mathcal{J}(\mathcal{H})$, $L > 0$, and Ω symplectic form on $[-L, L] \times K$ taming J which coincides with ω on $T(\{\pm L\} \times K)$, up to positive constants. Then there are neighborhoods \mathcal{O} of \mathcal{H} , \mathcal{D} of Ω , \mathcal{W} of ω in the C^∞ -topology (weak or strong), and \mathcal{U} of J in the strong C^∞ -topology such that if $\mathcal{H}' = (\xi', X', \omega') \in \mathcal{O}$, $\bar{\omega} \in \mathcal{W}$, $R > L$, $J' \in \mathcal{J}(\mathcal{H}') \cap \mathcal{U}$, $\tilde{J} \in \mathcal{J}_{L < R}(J') \cap \mathcal{U}$ and $\Omega'_\pm \in \mathcal{D}$ are symplectic forms on $[-L, L] \times N$ satisfying*

- Ω'_+ coincides with ω' on $T(L \times N)$ and with $\bar{\omega}$ on $T(-L \times N)$, up to positive constants;
- Ω'_- coincides with $\bar{\omega}$ on $T(L \times N)$ and with ω' on $T(-L \times N)$, up to positive constants;

and \tilde{J} is adjusted to $\bar{\omega}$ on the neck $[L - R, R - L] \times K$, see 2.2.2, then the following assertions hold.

- (1) $(\tau_{\mp R})^* \Omega'_\pm$ tames \tilde{J} on $(\pm[R - L, R + L]) \times K$.
- (2) There exists $C > 0$ such that every finite-energy \tilde{J} -holomorphic map $F = (a, f) : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times N$ satisfying
 - i) $f(\mathbb{R} \times S^1) \subset K$ and the loops $t \mapsto f(s, t)$ are homotopic to γ in K ;
 - ii) $\{+\infty\} \times S^1$ is a positive puncture and $\{-\infty\} \times S^1$ is a negative puncture of F ;
 must also satisfy $E(F) \leq C$ and $f(\mathbb{R} \times S^1) \subset \text{int}(K)$.

In ii) above we use the symplectic forms $(\tau_{\mp R})^* \Omega'_\pm$ on $(\pm[R - L, R + L]) \times K$ and the 2-form $\bar{\omega}$ on the neck $[L - R, R - L] \times K$ to define $E(F)$ as described in 2.2.3.

3.2. Special stable Hamiltonian structures.

Definition 3.6. We call $\mathcal{H} = (\xi, X, \omega)$ special near γ if there exists a small closed smooth tubular neighborhood K of γ as above such that one of the following (mutually excluding) conditions holds:

- i) \mathcal{H} is induced by some contact form, i.e., $\xi = \ker \alpha$, $X = X_\alpha$ and $\omega = C d\alpha$ where α is a contact form defined near K and $C > 0$ is a constant.
- ii) The 1-form λ given by (2.1) is closed on K .

If some K is given for which i) or ii) applies then we say \mathcal{H} is special for γ and K .

In order to define local contact homology of the pair (\mathcal{H}, γ) one needs to slightly perturb \mathcal{H} near K , within the class of stable hamiltonian structures, to make closed orbits inside K and homotopic to γ (in K) non-degenerate. This may not be possible in general. However, when \mathcal{H} is special near γ this perturbation can always be performed. This is well-known if i) holds. Assume \mathcal{H} falls into case ii). Let α be a primitive for ω on K , which exists since $H^2(K, \mathbb{R})$ vanishes. Possibly after replacing α by $C\lambda + \alpha$, with $C \gg 1$, we can assume $\inf_K i_X \alpha > 0$ and α is a contact form on K . The Reeb vector field X_α is a pointwise positive multiple of X . Let α' be a C^∞ -small perturbation of α near K so that all closed α' -Reeb orbits inside K homotopic to γ are non-degenerate. Then $\omega' = d\alpha'$ is a small perturbation of ω and $\mathcal{H}' = (\xi = \ker \lambda, X', \omega' = d\alpha')$, where the vector field X' is given by $i_{X'} \omega' = 0$ and $i_{X'} \lambda' = 1$, is a stable Hamiltonian structure C^∞ -close to \mathcal{H} . Moreover, since $\mathbb{R}X' = \mathbb{R}X_{\alpha'}$, closed X' -orbits inside K which are homotopic to γ (in K) are non-degenerate.

Remark 3.7. Consider $K = S^1 \times \overline{B}$ and a smooth Hamiltonian $H : K \rightarrow \mathbb{R}$ satisfying $dH_t(0) = 0, \forall t$. Assume 0 is an isolated 1-periodic orbit of the Hamiltonian vector field X_{H_t} characterized by $dH_t = i_{X_{H_t}}\omega_0$. The typical example of special stable hamiltonian structure satisfying ii) in Definition 3.6 is $\mathcal{H} = (\ker dt, \tilde{X}_H, \omega_H)$, where $\tilde{X}_H = \partial_t + X_{H_t}$ and $\omega_H = dH_t \wedge dt + \omega_0$. Then $x_0(t) = (t, 0)$ is an isolated 1-periodic orbit of \tilde{X}_H . In this case we can perturb H to obtain a non-degenerate perturbation of \mathcal{H} .

4. LOCAL CONTACT HOMOLOGY

Contact Homology was originally introduced in [9] inside the bigger framework of Symplectic Field Theory. Following Floer [11] we define a suitable version of what we call the local contact homology of an isolated orbit, see Definition 4.4.

4.1. Defining local contact homology.

4.1.1. Local chain complexes. Throughout Section 4 we fix $\mathcal{H} = (\xi, X, \omega)$, an isolated closed X -orbit $\gamma = (x, T)$, and assume \mathcal{H} is special for γ as in Definition 3.6. Since γ is isolated, we will fix an isolating neighborhood K of γ . Moreover, as explained in 3.2, perhaps after shrinking K , we can always perturb \mathcal{H} to an arbitrarily C^∞ -close $\mathcal{H}' = (\xi', X', \omega')$ so that all X' -orbits in K homotopic to γ are non-degenerate. We refer to \mathcal{H}' as a **non-degenerate perturbation of \mathcal{H}** . This notion depends on K and on the homotopy class of γ (in K).

Let $\mathcal{H}' = (\xi', X', \omega')$ be a small non-degenerate C^∞ -perturbation of \mathcal{H} . We denote by $\mathcal{P}(\mathcal{H}', K, \gamma)$ and $\mathcal{P}_0(\mathcal{H}', K, \gamma)$ the sets of closed X' -orbits and good closed X' -orbits in K which are homotopic to γ , respectively. By Lemma 3.3 every $\gamma' \in \mathcal{P}(\mathcal{H}', K, \gamma)$ is very close to γ if \mathcal{H}' is close enough to \mathcal{H} , in particular, they all lie in the interior of K . $\mathcal{P}(\mathcal{H}', K, \gamma)$ is finite if \mathcal{H}' is close enough to \mathcal{H} since there are automatic period bounds for the orbits in $\mathcal{P}(\mathcal{H}', K, \gamma)$. We fix a homotopy class of ω -symplectic trivializations of the bundle $x_T^*\xi \rightarrow \mathbb{R}/\mathbb{Z}$, where $x_T : \mathbb{R}/\mathbb{Z} \rightarrow N$ is the map given by $t \mapsto x(Tt)$. It distinguishes homotopy classes of ω' -symplectic trivializations of ξ' along every $\gamma' \in \mathcal{P}(\mathcal{H}', K, \gamma)$, which are used to compute Conley-Zehnder indices $\mu_{CZ}(\gamma')$. Let $C_*(\mathcal{H}', K, \gamma)$ be the vector space over \mathbb{Q} freely generated by $\mathcal{P}_0(\mathcal{H}', K, \gamma)$ and graded by $|\gamma'| = \mu_{CZ}(\gamma') + n - 3$.

In order to define a differential on $C_*(\mathcal{H}', K, \gamma)$ we need to choose $J \in \mathcal{J}(\mathcal{H})$ and assume that we can find $J' \in \mathcal{J}(\mathcal{H}')$ arbitrarily C^∞ -close to J (strong or weak) which is regular for the data $(\mathcal{H}', K, \gamma)$ in the following sense. Consider the set $\mathcal{F}(J', K, \gamma)$ of finite-energy J' -holomorphic maps $F : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times N$ with a positive (negative) puncture at $+\infty \times S^1$ (at $-\infty \times S^1$), with image in $\mathbb{R} \times K$, and asymptotic to orbits in $\mathcal{P}(\mathcal{H}', K, \gamma)$. Then we call J' regular for the data $(\mathcal{H}', K, \gamma)$ if the linearized Cauchy-Riemann equations at every $F \in \mathcal{F}(J', K, \gamma)$ determines a surjective Fredholm operator in a standard functional analytical set-up; see [33].

Remark 4.1. The above described transversality assumption does not hold in general. The transversality issues in Symplectic Field Theory are expected to be solved by the work of Hofer, Wysocki and Zehnder [22, 23, 24].

Let $\gamma', \gamma'' \in \mathcal{P}(\mathcal{H}', K, \gamma)$, and consider the moduli spaces $\mathcal{M}_{K, J'}(\gamma'; \gamma'')$ consisting of equivalence classes of triples (t^+, t^-, F) , where $t^\pm \in S^1$ and $F \in \mathcal{F}(J', K, \gamma)$ is asymptotic to γ' and γ'' at the positive and negative puncture, respectively. Moreover, writing $F = (a, f)$, it

is required that

$$(4.1) \quad \begin{aligned} \lim_{s \rightarrow +\infty} f(s, t^+) &= \text{pt}_{\gamma'} \\ \lim_{s \rightarrow -\infty} f(s, t^-) &= \text{pt}_{\gamma''}. \end{aligned}$$

Two triples (t^+, t^-, F) and (τ^+, τ^-, G) are equivalent if there exist $\Delta s, \Delta t \in \mathbb{R}$ such that $F(s, t) = G(s + \Delta s, t + \Delta t)$ and $t^\pm + \Delta t = \tau^\pm$. Under the above mentioned regularity assumption then, according to Theorem 0 from [33], we find that $\mathcal{M}_{K, J'}(\gamma'; \gamma'')$ is a smooth orbifold of dimension $|\gamma'| - |\gamma''|$. When $\gamma' \neq \gamma''$ these spaces are equipped with a free \mathbb{R} -action induced by translations in the first coordinate of the target manifold $\mathbb{R} \times N$.

The proof of the following statement will be deferred to the end of 4.1.1.

Lemma 4.2. *If \mathcal{H}' is a non-degenerate sufficiently C^∞ -small perturbation of \mathcal{H} , and $J' \in \mathcal{J}(\mathcal{H}')$ is sufficiently C^∞ -close to J and regular for the data $(\mathcal{H}', K, \gamma)$, then $\mathcal{M}_{K, J'}(\gamma'; \gamma'')/\mathbb{R}$ is finite, for any pair of orbits $\gamma', \gamma'' \in \mathcal{P}(\mathcal{H}', K, \gamma)$ satisfying $|\gamma'| = |\gamma''| + 1$.*

Under the above assumptions on \mathcal{H}' and J' we follow [4] and associate to every $[t^+, t^-, F] \in \mathcal{M}_{K, J'}(\gamma'; \gamma'')$ a sign $\epsilon[t^+, t^-, F] = \pm 1$ by using suitable coherent orientations of these moduli spaces (and comparing them with the orientation given by the infinitesimal \mathbb{R} -action). These signs are well-defined even when γ' or γ'' is a bad orbit. Following [9], we set

$$(4.2) \quad n(\gamma', \gamma'') = \sum_{[t^+, t^-, F] \in \mathcal{M}_{K, J'}(\gamma'; \gamma'')/\mathbb{R}} \epsilon[t^+, t^-, F]$$

for every pair $\gamma', \gamma'' \in \mathcal{P}_0(\mathcal{H}', K, \gamma)$ satisfying $|\gamma'| - |\gamma''| = 1$. Set $n(\gamma', \gamma'') = 0$ otherwise. Finally we define a linear map

$$(4.3) \quad d : C_*(\mathcal{H}', K, \gamma) \rightarrow C_{*-1}(\mathcal{H}', K, \gamma)$$

by

$$(4.4) \quad \gamma' \mapsto \sum_{\gamma'' \in \mathcal{P}_0(\mathcal{H}', K, \gamma)} \frac{n(\gamma', \gamma'')}{m_{\gamma''}} \gamma''$$

on generators $\gamma' \in \mathcal{P}_0(\mathcal{H}', K, \gamma)$. Note that the coefficients in the above sum are integers.

Lemma 4.3. *If \mathcal{H}' is a non-degenerate sufficiently C^∞ -small perturbation of \mathcal{H} , $J' \in \mathcal{J}(\mathcal{H}')$ is sufficiently C^∞ -close to J and regular for the data $(\mathcal{H}', K, \gamma)$, then the square of the map (4.3) vanishes.*

The proof of the above statement strongly relies on Lemma 3.4 and will be given below.

Definition 4.4. *Let γ be an isolated closed orbit for the special stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega)$, and take a small isolating neighborhood K for γ . Let \mathcal{H}' be a small non-degenerate perturbation of \mathcal{H} , and $J' \in \mathcal{J}(\mathcal{H}')$ be a small perturbation of J in the strong C^∞ -topology which is regular for the data $(\mathcal{H}', K, \gamma)$. The local contact homology of $HC(\mathcal{H}, \gamma)$ is defined as the homology of the complex $(C_*(\mathcal{H}', K, \gamma), d)$.*

The remaining of Section 4 is devoted to showing that this definition does not depend on the choices of K , \mathcal{H}' and J' with the above properties.

Proof of Lemma 4.2. The argument is standard. If the lemma does not hold we find a sequence $C_n \in \mathcal{M}_{K, J'}(\gamma'; \gamma'')/\mathbb{R}$ of distinct elements. Energy bounds for $\{C_n\}$ are automatically guaranteed by Lemma 3.4 since the data (\mathcal{H}', J') is assumed arbitrarily close to (\mathcal{H}, J) . The

limiting behavior of the sequence is described by the SFT-Compactness Theorem from [3]. Using a primitive for ω' on K one constructs an exact symplectic form on $\mathbb{R} \times K$ taming J' , so that the limiting holomorphic building of the sequence C_n does not contain spheres. Also, finite-energy J' -holomorphic punctured spheres on $\mathbb{R} \times K$ must have positive punctures. Since there are no contractible X' -orbits in K , Lemma A.1 and Lemma A.6 together imply that the limiting holomorphic building does not contain planes. Hence, it must be a broken cylinder with possibly many levels, all contained in $\mathbb{R} \times K$. However, by additivity of the Fredholm indices and regularity of J' there is only one level, which must be an element of $\mathcal{M}_{K,J'}(\gamma'; \gamma'')/\mathbb{R}$. But these are isolated, again by regularity, a contradiction. \square

Proof of Lemma 4.3. The proof follows a standard argument, see [2]. Consider closed X' -orbits $\gamma, \gamma'' \in \mathcal{P}(\mathcal{H}', K, \gamma)$ satisfying $|\gamma| - |\gamma''| = 2$. We need to show first that sequences of elements in $\mathcal{M}_{K,J'}(\gamma, \gamma'')/\mathbb{R}$ must necessarily converge to a 2-level holomorphic building, its upper level being an element of $\mathcal{M}_{K,J'}(\gamma; \gamma')/\mathbb{R}$ and its lower level belonging to $\mathcal{M}_{K,J'}(\gamma'; \gamma'')/\mathbb{R}$. This follows from an argument similar to that given in the proof of Lemma 4.2, using that there are no finite-energy J' -holomorphic spheres on $\mathbb{R} \times K$, using also that limiting holomorphic buildings of sequences in $\mathcal{M}_{K,J'}(\gamma, \gamma'')/\mathbb{R}$ do not contain planes, and that J' is regular for the relevant cylinders. Clearly, we used compactness of K and the automatic energy bounds from Lemma 3.4 since (\mathcal{H}', J') can be as close to (\mathcal{H}, J) as we want. Secondly, one must show that all such 2-level broken cylinders arise as SFT-limits of sequences in $\mathcal{M}_{K,J'}(\gamma, \gamma'')/\mathbb{R}$. To this end we argue that, since we allow (\mathcal{H}', J') to be as C^∞ -close to (\mathcal{H}, J) as we please, all maps $F = (a, f) : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times K$ representing elements of $\mathcal{M}_{K,J'}(\gamma^+; \gamma^-)$ satisfy $\overline{f(\mathbb{R} \times S^1)} \subset \text{int}(K)$, for any $\gamma^\pm \in \mathcal{P}(\mathcal{H}', K, \gamma)$. This follows from an application of Lemma 3.4. Therefore, using the assumed regularity, we can glue a cylinder in $\mathcal{M}_{K,J'}(\gamma; \gamma')/\mathbb{R}$ with a cylinder in $\mathcal{M}_{K,J'}(\gamma'; \gamma'')/\mathbb{R}$ and again obtain a cylinder in $\mathcal{M}_{K,J'}(\gamma; \gamma'')/\mathbb{R}$ (the projection of the image of the glued cylinder still lies in $\text{int}(K)$). Thus d^2 counts algebraically boundary points of entire 1-dimensional moduli spaces (broken cylinders with a bad orbit in between cancel each other). \square

4.1.2. Chain maps. The first step to prove that Definition 4.4 is well-posed is to define suitable chain maps between the chain complexes induced by different perturbations. We consider C^∞ -small non-degenerate perturbations $\mathcal{H}' = (\xi', X', \omega')$, $\mathcal{H}'' = (\xi'', X'', \omega'')$ of \mathcal{H} , as explained in 4.1.1. Also, we select $J' \in \mathcal{J}(\mathcal{H}')$ and $J'' \in \mathcal{J}(\mathcal{H}'')$ C^∞ -close to J and regular for the data $(\mathcal{H}', K, \gamma)$ and $(\mathcal{H}'', K, \gamma)$, respectively.

We assumed \mathcal{H} is special for γ . Consequently, according to Definition 3.6, it is either induced by some contact form α , or the 1-form λ as in (2.1) is closed. In the first case we consider $\Omega_0 = d(e^a \alpha)$, and in the second case consider $\Omega_0 = d(Ae^a \lambda + \alpha)$ where α is some primitive of ω on K , $A \gg 1$ and a is the \mathbb{R} -coordinate. In both cases J is Ω_0 -compatible. For any $L > 0$ we can find a small exact perturbation Ω of Ω_0 on $[-L, L] \times K$, which agrees with a positive multiple of ω' on $T(\{L\} \times K)$ and with a positive multiple of ω'' on $T(\{-L\} \times K)$. For any fixed $L > 0$ we may find an almost complex structure $\bar{J} \in \mathcal{J}_L(J'', J')$. Taking J', J'' sufficiently C^∞ -close to J , we can find such \bar{J} arbitrarily close to J in the C^∞ -strong topology. Then \bar{J} will be Ω -tamed when Ω is a small perturbation of Ω_0 as described above. Ω will be used to define energy of \bar{J} -holomorphic maps.

Analogously as before, consider the space $\mathcal{F}(\bar{J}, K, \gamma)$ of finite-energy \bar{J} -holomorphic maps $F = (a, f) : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times N$, with image in $\mathbb{R} \times K$ and in the homotopy class of γ (meaning that $t \mapsto f(s, t)$ is homotopic to γ in K), see 4.1.1, with a positive (negative) puncture

at $+\infty \times S^1$ (at $-\infty \times S^1$). We need again to assume that regularity is achieved for such cylinders by arbitrarily small perturbations of \bar{J} within $\mathcal{J}_L(J'', J')$. After such perturbation, in the appropriate functional analytical set-up, the linearized Cauchy-Riemann equations at all $F \in \mathcal{F}(\bar{J}, K, \gamma)$ determine surjective Fredholm operators. In this case we call \bar{J} regular for the data $((\mathcal{H}', J'), (\mathcal{H}'', J''), K, \gamma)$.

Remark 4.5. *Such transversality assumptions are not expected to hold in general, and one needs the difficult analytical tools from [22, 23, 24] in order to achieve transversality in a suitable sense.*

Given any $\gamma' \in \mathcal{P}(\mathcal{H}', K, \gamma)$ and $\gamma'' \in \mathcal{P}(\mathcal{H}'', K, \gamma)$, we consider the space $\mathcal{M}_{K, \bar{J}}(\gamma'; \gamma'')$ of equivalence classes of triples (t^+, t^-, F) , where $t^\pm \in S^1$ and $F \in \mathcal{F}(\bar{J}, K, \gamma)$ is asymptotic to γ' and γ'' at the positive and negative puncture, respectively. The equivalence relation is exactly as discussed in 4.1.1. Under the above transversality assumption, this is a smooth orbifold of dimension $|\gamma'| - |\gamma''|$; see Theorem 0 in [33]. If $|\gamma'| - |\gamma''| = 0$ one can associate signs $\epsilon[t^+, t^-, F]$ to classes $[t^+, t^-, F] \in \mathcal{M}_{K, \bar{J}}(\gamma'; \gamma'')$: these are 0-dimensional and one has the coherent orientations obtained from [4]. Again, there is no need to assume γ' or γ'' are good.

Lemma 4.6. *If the data $(\mathcal{H}', J'), (\mathcal{H}'', J'')$ with the above properties is sufficiently close to (\mathcal{H}, J) and \bar{J} is a small C^∞ -perturbation of J which is regular for the data $((\mathcal{H}', J'), (\mathcal{H}'', J''), K, \gamma)$, then the moduli space $\mathcal{M}_{K, \bar{J}}(\gamma'; \gamma'')$ is finite when $|\gamma'| - |\gamma''| = 0$.*

Proof. The proof of the above statement is entirely analogous to that of Lemma 4.2. Only note that one can construct an exact symplectic form on $\mathbb{R} \times K$ taming \bar{J} . For that one uses primitives of ω', ω'' on K which are C^∞ -close to a given primitive of ω . There are automatic energy bounds for sequences in $\mathcal{M}_{K, \bar{J}}(\gamma'; \gamma'')$, by Lemma 3.4. Hence limiting holomorphic buildings of such sequences contain no spheres. Since there are no finite-energy pseudo-holomorphic planes with respect to J', J'' or \bar{J} in $\mathbb{R} \times K$ (X' and X'' have no contractible orbits in K , see Lemma A.6), any limiting holomorphic building of a sequence in $\mathcal{M}_{K, \bar{J}}(\gamma'; \gamma'')$ must be a broken cylinder with possibly many levels. An index argument concludes the proof since we assume regularity for all relevant cylinders. \square

Analogously to [9] we set

$$n(\gamma', \gamma'') = \sum_{[t^+, t^-, F] \in \mathcal{M}_{K, \bar{J}}(\gamma'; \gamma'')} \epsilon[t^+, t^-, F]$$

if $|\gamma'| - |\gamma''| = 0$, or $n(\gamma', \gamma'') = 0$ if $|\gamma'| - |\gamma''| \neq 0$. Then there is a degree 0 map

$$(4.5) \quad \Phi : C_*(\mathcal{H}', K, \gamma) \rightarrow C_*(\mathcal{H}'', K, \gamma)$$

defined by

$$(4.6) \quad \gamma' \mapsto \sum_{\gamma'' \in \mathcal{P}_0(\mathcal{H}'', K, \gamma)} \frac{n(\gamma', \gamma'')}{m_{\gamma''}} \gamma''.$$

As in (4.4), the coefficients in (4.6) are integers.

Lemma 4.7. *If the data $(\mathcal{H}', J'), (\mathcal{H}'', J'')$ as above are sufficiently close to (\mathcal{H}, J) and \bar{J} is sufficiently close to J , then the map (4.5) is a chain map with respect to the differentials defined in (4.3).*

Proof. The argument is entirely analogous to that for Lemma 4.3. Again the relevant facts are: a) there are no pseudo-holomorphic spheres or finite-energy planes in $\mathbb{R} \times K$ with respect J' , J'' or \bar{J} ; b) the closures of the images of the projections onto N of all cylinders in $\mathcal{F}(J', K, \gamma)$, $\mathcal{F}(J'', K, \gamma)$ or $\mathcal{F}(\bar{J}, K, \gamma)$ are strictly contained in $\text{int}(K)$. Note that b) is achieved by an application of Lemma 3.4 since J' , J'' and \bar{J} are allowed to be taken arbitrarily close to J . Lemma 3.4 also provides automatic energy bounds for these cylinders (note our careful choices of taming symplectic forms on $[-L, L] \times K$). Let us fix $\gamma \in \mathcal{P}(\mathcal{H}', K, \gamma)$ and $\gamma'' \in \mathcal{P}(\mathcal{H}'', K, \gamma)$ satisfying $|\gamma| - |\gamma''| = 1$. By a), the automatic energy bounds and the assumed regularity for cylinders, SFT-limits of sequences in $\mathcal{M}_{K, \bar{J}}(\gamma; \gamma'')$ must be a holomorphic building of height-2 with two cylindrical levels, one of them is \bar{J} -holomorphic and the other is holomorphic with respect to either J' or J'' . The upper level is asymptotic to γ at its positive puncture, and the lower level is asymptotic to γ'' at its negative puncture. Now b) implies that the height-2 holomorphic buildings as just described arise as limits of a sequence in $\mathcal{M}_{K, \bar{J}}(\gamma; \gamma'')$, since after glueing both levels we must obtain a cylinder with image inside $\mathbb{R} \times K$. \square

4.1.3. *Homotopies.* Two chain maps as in (4.5) turn out to be chain homotopic. To see this, consider $\mathcal{H}', \mathcal{H}''$ small non-degenerate perturbations of \mathcal{H} . Consider also $J' \in \mathcal{J}(\mathcal{H}')$, $J'' \in \mathcal{J}(\mathcal{H}'')$ small perturbations of $J \in \mathcal{J}(\mathcal{H})$ which are regular for the data $(\mathcal{H}', K, \gamma)$ and $(\mathcal{H}'', K, \gamma)$, respectively, and to which the conclusions of Lemma 3.4 apply.

Let $\bar{J}_0, \bar{J}_1 \in \mathcal{J}_L(J'', J')$ be regular for the data $((\mathcal{H}', J'), (\mathcal{H}'', J''), K, \gamma)$, and C^∞ -close to J with the properties explained in 4.1.2. These exist since J', J'' are allowed to be taken very C^∞ -close to J . Then we may find paths $\{\bar{J}_t\} \in \mathcal{J}_{\tau, L}(J'', J')$ connecting \bar{J}_0 to \bar{J}_1 which lie on a arbitrarily given small C^∞ -strong neighborhood of J , provided that \bar{J}_0, \bar{J}_1 are sufficiently C^∞ -close to J . Moreover, one also finds an exact symplectic form Ω on $[-L, L] \times K$ C^∞ -close to Ω_0 as described in 4.1.2, which tames all \bar{J}_t , is equal to a positive constant multiple of ω'' on $-L \times N$, and to a positive constant multiple of ω' on $L \times N$. Such a symplectic form can be used to define the energy of \bar{J}_t -holomorphic maps, for all $t \in [0, 1]$.

We need the path $\{\bar{J}_t\}$ to be regular for the data $((\mathcal{H}', J'), (\mathcal{H}'', J''), K, \gamma)$ in the following sense. Let $\mathcal{F}(\{\bar{J}_t\}, K, \gamma)$ denote the set of pairs (t, F) , where $t \in [0, 1]$, and $F : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times K$ is a finite-energy \bar{J}_t -holomorphic cylinder with a positive (negative) puncture at $+\infty \times S^1$ (at $-\infty \times S^1$), asymptotic to an orbit in $\mathcal{P}(\mathcal{H}', K, \gamma)$ at the positive puncture and to an orbit in $\mathcal{P}(\mathcal{H}'', K, \gamma)$ at the negative puncture. Regularity of $\{\bar{J}_t\}$ means that, in a standard functional analytical set-up, the linearization of the $(t$ -dependent) Cauchy-Riemann equations at every $(t, F) \in \mathcal{F}(\{\bar{J}_t\}, K, \gamma)$ is surjective. We will assume that any path in $\mathcal{J}_{\tau, L}(J'', J')$ can be slightly C^∞ -perturbed to a regular path. Note that since \bar{J}_0, \bar{J}_1 are already assumed regular then the perturbation may be done keeping the endpoints fixed.

Remark 4.8. *The above mentioned regularity may not always be obtained. One requires analytical tools from [22, 23, 24] to describe the precise set-up where the appropriate notion of transversality can be achieved.*

Consider $\{\bar{J}_t\}$ regular for the data $((\mathcal{H}', J'), (\mathcal{H}'', J''), K, \gamma)$ and contained in a small C^∞ -neighborhood of J . For fixed $\gamma' \in \mathcal{P}(\mathcal{H}', K, \gamma)$ and $\gamma'' \in \mathcal{P}(\mathcal{H}'', K, \gamma)$, consider the moduli space $\mathcal{M}_{K, \{\bar{J}_t\}}(\gamma'; \gamma'')$ of pairs $(t, [t^+, t^-, F])$, where $(t, F) \in \mathcal{F}(\{\bar{J}_t\}, K, \gamma)$ and $[t^+, t^-, F] \in \mathcal{M}_{K, \bar{J}_t}(\gamma'; \gamma'')$. Here the space $\mathcal{M}_{K, \bar{J}_t}(\gamma'; \gamma'')$ was defined in 4.1.2. Under our regularity assumption the space $\mathcal{M}_{K, \{\bar{J}_t\}}(\gamma'; \gamma'')$ is a smooth orbifold with boundary of dimension $|\gamma'| - |\gamma''| + 1$.

Lemma 4.9. *Under the above hypotheses, the moduli space $\mathcal{M}_{K,\{\bar{J}_t\}}(\gamma'; \gamma'')$ is finite when $|\gamma'| - |\gamma''| + 1 = 0$.*

The proof is entirely analogous to those of Lemma 4.2 or Lemma 4.6 and will be omitted. We need automatic energy bounds for elements in $\mathcal{F}(\{\bar{J}_t\}, K, \gamma)$, which is achieved by Lemma 3.4 in view of the special form of our small perturbations of the data (\mathcal{H}, J) and by the properties of Ω .

As before, when $|\gamma'| - |\gamma''| + 1 = 0$ there are signs $\epsilon(t, [t^+, t^- F])$ associated to each element $(t, [t^+, t^- F])$ of $\mathcal{M}_{K,\{\bar{J}_t\}}(\gamma'; \gamma'')$ induced by a system of orientations which is coherent with the glueing operation; see [4].

We set

$$(4.7) \quad n(\gamma', \gamma'') = \sum_{(t, [t^+, t^-, F]) \in \mathcal{M}_{K,\{\bar{J}_t\}}(\gamma', \gamma'')} \epsilon(t, [t^+, t^-, F])$$

when $|\gamma'| - |\gamma''| = -1$, or $n(\gamma', \gamma'') = 0$ when $|\gamma'| - |\gamma''| \neq -1$. There is a degree +1 map

$$(4.8) \quad T : C_*(\mathcal{H}', K, \gamma) \rightarrow C_{*+1}(\mathcal{H}'', K, \gamma)$$

defined by

$$(4.9) \quad T\gamma' = \sum_{\gamma'' \in \mathcal{P}_0(K, \mathcal{H}'', \gamma)} \frac{n(\gamma', \gamma'')}{m_{\gamma''}} \gamma''$$

on generators.

Lemma 4.10. *Let $\Phi_j : C_*(\mathcal{H}', K, \gamma) \rightarrow C_*(\mathcal{H}'', K, \gamma)$ be the chain map (4.5) induced by \bar{J}_j , $j = 0, 1$. Then $\Phi_0 - \Phi_1 = \pm(T \circ d - d \circ T)$.*

The argument is analogous as the ones given to prove lemmas 4.3 and 4.7, only note here that moduli spaces $\mathcal{M}_{K,\{\bar{J}_t\}}(\gamma; \gamma'')$ with $\gamma \in \mathcal{P}(\mathcal{H}', K, \gamma)$ and $\gamma'' \in \mathcal{P}(\mathcal{H}'', K, \gamma)$ satisfying $|\gamma| = |\gamma''|$ do have a genuine boundary, corresponding to elements of $\mathcal{M}_{K,\bar{J}_0}(\gamma; \gamma'')$ and $\mathcal{M}_{K,\bar{J}_1}(\gamma; \gamma'')$. As before one needs to make strong use of Lemma 3.4 to conclude that certain glued cylinders lie in $\mathbb{R} \times K$.

4.1.4. Stability of local contact homology. Here we study the chain maps (4.5) more closely and show that they induce isomorphisms between the homologies of the local chain complexes.

Consider, as before, pairs (\mathcal{H}', J') , (\mathcal{H}'', J'') , where $\mathcal{H}' = (\xi', X', \omega')$ and $\mathcal{H}'' = (\xi'', X'', \omega'')$ are small special non-degenerate perturbations of \mathcal{H} , and $J' \in \mathcal{J}(\mathcal{H}')$, $J'' \in \mathcal{J}(\mathcal{H}'')$ are small perturbations of J which are regular for the data $(\mathcal{H}', K, \gamma)$ and $(\mathcal{H}'', K, \gamma)$, respectively. We can assume that the conclusions of Lemma 3.3 hold for $\mathcal{H}', \mathcal{H}''$, and those of Lemma 3.4 hold for J', J'' . We fix $L > 0$ and choose $\bar{J}^+ \in \mathcal{J}_L(J'', J')$ and $\bar{J}^- \in \mathcal{J}_L(J', J'')$ sufficiently small perturbations of J in the strong C^∞ -topology, to which the conclusions of Lemma 3.4 also apply. This can be achieved since we have the freedom of choosing J', J'' as C^∞ -close to J as we want. We also assume that \bar{J}^\pm are regular, as explained in 4.1.2. The energy of \bar{J}^\pm -holomorphic maps are defined using symplectic forms on $[-L, L] \times K$, see 4.1.2.

For any $R > L$ we can consider the almost complex structure \bar{J}_R defined by

$$(4.10) \quad \bar{J}_R = \begin{cases} (\tau_{-R})^* \bar{J}^+ & \text{on } [0, +\infty) \times N \\ (\tau_R)^* \bar{J}^- & \text{on } (-\infty, 0] \times N. \end{cases}$$

Note that $\bar{J}_R \in \mathcal{J}_{L < R}(J') \subset \mathcal{J}_{R+L}(J', J'')$, and if $J', J'', \bar{J}^+, \bar{J}^-$ are sufficiently close to J then \bar{J}_R lies in any given neighborhood of J in the C^∞ -strong topology, uniformly in $R > L$. As

is well-known for all choices $\gamma'_\pm \in \mathcal{P}(\mathcal{H}', K, \gamma)$ and $\gamma'' \in \mathcal{P}(\mathcal{H}'', K, \gamma)$ satisfying $|\gamma'_+| = |\gamma'_-| = |\gamma''|$, when R is large enough there is a surjective glueing map

$$(4.11) \quad \#_R : \mathcal{M}_{K, \bar{J}^+}(\gamma'_+; \gamma'') \times \mathcal{M}_{K, \bar{J}^-}(\gamma''; \gamma'_-) \rightarrow \mathcal{M}_{K, \bar{J}_R}(\gamma'_+; \gamma'_-).$$

Regularity is crucial in order to get this map well-defined. Also, finite-energy pseudo-holomorphic cylinders in K with respect to \bar{J}^\pm connecting orbits homotopic to γ (in K) have the closure of the projection of their images onto K contained in the interior of K , by Lemma 3.4. So the same holds for the glued cylinders, and this is the reason why they lie in $\mathcal{M}_{K, \bar{J}_R}(\gamma'_+; \gamma'_-)$. How the cylinders are glued is determined by the asymptotic markers at the punctures corresponding to the orbit γ'' , and different configurations could induce the same glued cylinder. So, this map is not injective and the same glued cylinder appears $m_{\gamma''}$ times. Thus, for fixed γ'_\pm and γ'' as above with $|\gamma'_+| = |\gamma'_-| = |\gamma''|$, we have the formula

$$(4.12) \quad \begin{aligned} & \frac{1}{m_{\gamma''}} \sum_{\substack{[t^+, t^-, F] \in \mathcal{M}_{K, \bar{J}^+}(\gamma'_+; \gamma'') \\ [\tau^+, \tau^-, G] \in \mathcal{M}_{K, \bar{J}^-}(\gamma''; \gamma'_-)}} \epsilon[t^+, t^-, F] \epsilon[\tau^+, \tau^-, G] \\ &= \sum_{[\theta^+, \theta^-, H] \in \mathcal{M}_{K, \bar{J}_R}(\gamma'_+; \gamma'_-)} \epsilon[\theta^+, \theta^-, H]. \end{aligned}$$

The fact that the orientations are coherent under glueing was used above. It follows from (4.12) that

$$(4.13) \quad \Phi_- \circ \Phi_+ = \Phi_R$$

where Φ_\pm are the chain maps (4.5) induced by \bar{J}^\pm , and Φ_R is the chain map induced by \bar{J}_R .

As explained above, the glued cylinders have images contained in $\mathbb{R} \times \text{int}(K)$. The glueing analysis will give surjectivity for the linearized Cauchy-Riemann operators at the maps parametrizing these glued cylinders, so that \bar{J}_R is regular for the data $((\mathcal{H}', J'), (\mathcal{H}'', J''), K, \gamma)$ in the sense explained in 4.1.2. Recall that \bar{J}_R can be arranged to lie on a small neighborhood of J (in the C^∞ -strong topology).

Note that $\bar{J}_R \in \mathcal{J}_{L < R}(J')$ is adjusted to ω'' on the neck $[L - R, R - L] \times K$. Moreover, one can find regular⁴ homotopies $\{\bar{J}_t\} \subset \mathcal{J}_{L < R}(J')$ connecting \bar{J}_R to J' inside an arbitrarily small neighborhood of J in the strong C^∞ -topology, since we are allowed to assume $J', J'' \bar{J}^+, \bar{J}^-$ are arbitrarily close to J . Moreover, the convex combination $\omega_t := (1 - t)\omega'' + t\omega'$ is a path of closed 2-forms near K of maximal rank (since $\omega'' \sim \omega'$) and the path \bar{J}_t can be arranged to be adjusted to ω_t on the neck $[L - R, R - L] \times K$. We can apply Lemma 3.5 to conclude that finite-energy \bar{J}_R cylinders $F = (a, f)$ in $\mathbb{R} \times K$ connecting orbits homotopic to γ in K satisfy $\overline{f(\mathbb{R} \times S^1)} \subset \text{int}(K)$. Note that we have automatic energy bounds for such cylinders, and that the conclusion we obtained is independent of $R > L$.

Thus, we can argue as previously explained in 4.1.3 to get a chain homotopy between the chain map Φ_R and the chain map induced by the \mathbb{R} -invariant J' . This last chain map induces the identity already at the chain level. In view of (4.13) we conclude that $\Phi_- \circ \Phi_+$ is the identity at the homology level. We proved

Lemma 4.11. *Suppose that $\mathcal{H}', \mathcal{H}''$ are sufficiently small special non-degenerate C^∞ -perturbations of \mathcal{H} . Suppose also that $J' \in \mathcal{J}(\mathcal{H}')$ and $J'' \in \mathcal{J}(\mathcal{H}'')$ are sufficiently C^∞ -close to J*

⁴Regularity here needs to be assumed.

and regular for the data $(\mathcal{H}', K, \gamma)$ and $(\mathcal{H}'', K, \gamma)$, respectively. Then the homologies of $(C_*(\mathcal{H}', K, \gamma), d)$ and $(C_*(\mathcal{H}'', K, \gamma), d)$ defined above are isomorphic.

It follows from our discussion that there are well-defined graded vector spaces $HC_*(\mathcal{H}, K, \gamma, J)$ given by the homology of the chain complex $(C_*(\mathcal{H}', K, \gamma), d)$ where K is a small tubular neighborhood of γ and the data $(\mathcal{H}', J') \simeq (\mathcal{H}, J)$ is carefully chosen as above. We still need to address the independence of $HC_*(\mathcal{H}, K, \gamma, J)$ on J and K , which will be done below.

4.2. Invariance of local contact homology.

Lemma 4.12. *Let $\{\mathcal{H}^s = (\xi^s, X^s, \omega^s)\}_{s \in [0,1]}$ be a smooth family of stable hamiltonian structures on a manifold N , and $J^s \in \mathcal{J}(\mathcal{H}^s)$ be a smooth 1-parameter family of \mathbb{R} -invariant almost complex structures. Let γ be a closed X^0 -orbit and let K be a small compact tubular neighborhood of (the geometric image of) γ such that for every $s \in [0, 1]$ the following hold:*

- (a) *the vector field X^s is a pointwise positive multiple of X^0 on the geometric image of γ ,*
- (b) *γ is the only closed orbit of X^s contained in K in its free homotopy class (of loops in K),*
- (c) *X^s has no closed orbit contained in K which is contractible in K ,*
- (d) *Either \mathcal{H}^s is induced by some contact form on K , or the 1-form λ^s associated to \mathcal{H}^s as in (2.1) is closed on K (see Definition 3.6).*

Then $HC_(\mathcal{H}^0, K, \gamma, J^0) \simeq HC_*(\mathcal{H}^1, K, \gamma, J^1)$.*

In (b) above we abuse the notation and see γ as a closed X^s -orbit. This is possible in view of (a).

Proof. It is an immediate consequence of Lemma 4.11 that for every $s_0 \in [0, 1]$ there exists $\epsilon > 0$ such that $HC_*(\mathcal{H}^s, K, \gamma, J^s) = HC_*(\mathcal{H}^{s_0}, K, \gamma, J^{s_0})$ for all $s \in [0, 1]$ satisfying $|s - s_0| < \epsilon$. In fact, if not, we find a sequence $s_n \rightarrow s_0$ such that $HC_*(\mathcal{H}^{s_n}, K, \gamma, J^{s_n}) \neq HC_*(\mathcal{H}^{s_0}, K, \gamma, J^{s_0})$, $\forall n$. By our transversality assumptions, there are very small C^∞ -perturbations (\mathcal{H}'_n, J'_n) of $(\mathcal{H}^{s_n}, J^{s_n})$ such that \mathcal{H}'_n is non-degenerate, J'_n is regular for the data $(\mathcal{H}'_n, K, \gamma)$, $(\mathcal{H}'_n, J'_n) \rightarrow (\mathcal{H}^{s_0}, J^{s_0})$ in C^∞ as $n \rightarrow \infty$, and the conclusions of Lemma 3.4 hold for all J'_n . Moreover, the homology of the chain complex $(C_*(\mathcal{H}'_n, K, \gamma), d)$, where d is defined using J'_n , is $HC_*(\mathcal{H}^{s_n}, K, \gamma, J^{s_n})$. However, Lemma 4.11 says that these homology groups are also equal $HC_*(\mathcal{H}^{s_0}, K, \gamma, J^{s_0})$ when n is large, a contradiction. The conclusion now follows from compactness of $[0, 1]$. \square

As a consequence we can drop the dependence on J of the local contact homology of the data $(\mathcal{H}, K, \gamma, J)$. It is easy to see that it is also independent of the small tubular neighborhood K where γ is the only closed Hamiltonian orbit in its free homotopy class (of loops in K). We will write simply $HC(\mathcal{H}, \gamma)$.

5. LOCAL CONTACT HOMOLOGY OF ISOLATED PRIME REEB ORBITS

In this section we establish the relation between local contact homology of an isolated prime Reeb orbit and the associated Poincaré return map to a local cross section.

Proposition 5.1. *Let α be a contact form on a manifold N , and γ be an isolated prime Reeb orbit. Let $\Sigma \subset N$ be an embedded hypersurface transverse to γ at a point $p \in \gamma$, so that the local first return map $\varphi : (U, p) \rightarrow (\Sigma, p)$ is well-defined on a small neighborhood U of p in Σ . Then $HC(\alpha, \gamma)$ and $HF(\varphi, p)$ are isomorphic.*

In the above statement we denote by $HF(\varphi, p)$ the local Floer homology at the isolated fixed point p of the germ of symplectic diffeomorphism φ of the symplectic manifold $(\Sigma, d\alpha|_\Sigma)$. The isomorphism in Proposition 5.1 is defined only up to an even shift in the grading, since the grading of the local Floer homology of a germ of Hamiltonian diffeomorphism near an isolated fixed point is only defined up to an even shift, see [16].

5.1. Local models.

Lemma 5.2. *Let α be a contact form on a $(2n-1)$ -dimensional manifold, and $\gamma = (x, T)$ be a prime closed α -Reeb orbit. Then there exists a tubular neighborhood $K \simeq \mathbb{R}/\mathbb{Z} \times \overline{B}$ of $x(\mathbb{R})$, where $B \subset \mathbb{R}^{2n-2}$ is a small open ball centered at the origin, with coordinates $(t, q_1, \dots, p_1, \dots)$, such that $x(\mathbb{R}) \simeq \mathbb{R}/\mathbb{Z} \times 0$, $\alpha \simeq Hdt + \lambda_0$, where $H : K \rightarrow \mathbb{R}$ satisfies $H_t(0) = T$, $dH_t(0) = 0$, and $\lambda_0 = \frac{1}{2} \sum_{k=1}^{n-1} q_k dp_k - p_k dq_k$.*

Proof. First, it is simple to get a tubular neighborhood $K \simeq \mathbb{R}/\mathbb{Z} \times \overline{B}$ such that $\alpha|_{\mathbb{R}/\mathbb{Z} \times 0} = Tdt$ and $d\alpha$ restricted to $0 \times \mathbb{R}^{2n-2} \subset T_{(t,0)}(\mathbb{R}/\mathbb{Z} \times B)$ coincides with ω_0 , $\forall t \in \mathbb{R}/\mathbb{Z}$. Now, by a parametrized version of Darboux's Theorem for symplectic forms, we can change coordinates to obtain $d\alpha|_{T(t \times \overline{B})} = \omega_0$, $\forall t$.

Now, let the α -Reeb flow be denoted by ϕ_t . On a small neighborhood U of $0 \in \mathbb{R}^{2n-2}$ we find a smooth function $\tau : [0, 1] \times U \rightarrow \mathbb{R}$ such that $\phi_{\tau(t,z)}(0, z) \in t \times \overline{B}$. The maps $\varphi_t(z) = \phi_{\tau(t,z)}(0, z)$ defined on U are symplectic embeddings fixing the origin. Hence, we can find a smooth Hamiltonian H_t defined near 0 such that φ_t is its Hamiltonian flow. Moreover, H_t can be arranged to be 1-periodic on t and, consequently, defines a smooth function near $\mathbb{R}/\mathbb{Z} \times 0$. There is no loss of generality to assume that $H_t(0) = T$. It must satisfy $dH_t(0) = 0$ since 0 is left fixed. Consider the vector field $\tilde{X}_H = \partial_t + X_{H_t}$, where $dH_t = i_{X_{H_t}}\omega_0$. By the definition of φ_t we get $i_{\tilde{X}_H}d\alpha = 0$. But our coordinates obtained so far guarantee that $d\alpha = \beta_t \wedge dt + \omega_0$, for some 1-periodic smooth family of 1-forms β_t defined near $0 \in \mathbb{R}^{2n-2}$. Consequently

$$0 = i_{\tilde{X}_H}d\alpha = (i_{X_{H_t}}\beta_t)dt - \beta_t + i_{X_{H_t}}\omega_0 = (i_{X_{H_t}}\beta_t)dt - \beta_t + dH_t$$

proving that $\beta_t = dH_t$. In other words, $d\alpha = dH_t \wedge dt + \omega_0$.

Let $\alpha_1 = Hdt + \lambda_0$, so that $d(\alpha - \alpha_1) = 0$. Moreover, $\int_{\mathbb{R}/\mathbb{Z} \times 0} \alpha - \alpha_1 = 0$ and, consequently, we find a smooth function f defined near $\mathbb{R}/\mathbb{Z} \times 0$ such that $df = \alpha - \alpha_1$. After subtracting a constant we can assume $f = 0$ on $\mathbb{R}/\mathbb{Z} \times 0$. Consider $\alpha_s = (1-s)\alpha + s\alpha_1$ and the vector field $Y_s = fX_{\alpha_s}$ where, for each $s \in [0, 1]$, X_{α_s} is the Reeb vector of the contact form α_s . Denoting by ψ_s the flow of Y_s we get

$$\frac{d}{ds}\psi_s^*\alpha_s = \psi_s^*(i_{Y_s}d\alpha_s + d(i_{Y_s}\alpha_s) + \alpha_1 - \alpha) = \psi_s^*(df + \alpha_1 - \alpha) = 0.$$

Moreover, $\mathbb{R}/\mathbb{Z} \times 0$ is left fixed by ψ_s . Using ψ_1 we obtain the desired coordinates. \square

5.2. Proof of Proposition 5.1. Let $\gamma = (x, T)$ be a prime closed isolated Reeb orbit for a contact form α , as in the statement of Proposition 5.1. In view of Lemma 5.2 we work on $K = \mathbb{R}/\mathbb{Z} \times \overline{B}$ with coordinates $(t, q_1, \dots, p_1, \dots)$, and assume $\alpha = H_t dt + \lambda_0$, $H_t(0) = T$, $dH_t(0) = 0$, and $x(t) = (t/T, 0)$. Also, we assume γ is the only closed α -Reeb orbit which goes once around the tube. Thus we can take $(\Sigma, d\alpha) = (0 \times B, \omega_0)$.

The 1-forms $\alpha_s = (1-s)\alpha + sdt$, $s \in [0, 1]$, are contact forms for $s < 1$ and $d\alpha_s = (1-s)d\alpha = (1-s)\omega_H$, where $\omega_H = dH_t \wedge dt + \omega_0$. Consequently the Reeb vector fields X_{α_s} , $s \in [0, 1)$, are

all positive multiples of each other, proving that $x(\mathbb{R})$ is the only closed α_s -Reeb orbit going once around the tube. Consider the family

$$\mathcal{H}_s = (\xi_s = \ker \alpha_s, X_{\alpha_s}, d\alpha = \omega_H), \quad s \in [0, 1]$$

of stable Hamiltonian structures. It can be smoothly continued to $[0, 1]$ by setting

$$\mathcal{H}_1 = (\xi_1 = \ker dt, \tilde{X}_H, \omega_H)$$

where $\tilde{X}_H = \partial_t + X_{H_t}$. Since the 2-form is independent of s , the conditions of Lemma 4.12 are fulfilled, so that $HC_*(\mathcal{H}_s, \gamma)$ does not depend on $s \in [0, 1]$. It is easy to check that $HC_*(\mathcal{H}_1, \gamma)$ coincides with the local Floer homology of the isolated 1-periodic orbit 0 of the Hamiltonian H_t , up to an even shift in the grading since the homotopy class of $d\alpha$ -symplectic trivializations along γ induced by the choice of coordinates given by Lemma 5.2 was not specified. This concludes the argument.

6. ESTIMATING LOCAL CONTACT HOMOLOGY

In this section we prove the following statement.

Proposition 6.1. *Let α be a contact form on a manifold N and $\gamma = (x, T = mT_0)$ be an isolated α -Reeb orbit with multiplicity m and minimal period $T_0 > 0$. Let $\Sigma \subset N$ be an embedded hypersurface transverse to γ at $p = x(0)$, so that the local first return map $\psi : (U, p) \rightarrow (\Sigma, p)$ is well-defined on a small neighborhood U of p in Σ . Then $\dim HC_*(\alpha, \gamma) \leq \dim HF_*(\psi^m, p)$, for every $*$ in \mathbb{Z} .*

The gradings in $HC_*(\alpha, \gamma)$ and in $HF_*(\psi^m, p)$ are given by the Conley-Zehnder indices computed with respect to homotopy classes of symplectic trivializations induced by a common homotopy class of $d\alpha$ -symplectic trivializations of $\xi = \ker \alpha$ along γ , which we fix from now on.

6.1. Geometric set-up and notation. Let n be defined by $\dim N = 2n - 1$, denote the Reeb vector field of α by R and fix $J \in \mathcal{J}(\alpha)$. Let $K \simeq \mathbb{R}/\mathbb{Z} \times \bar{B}$ be an isolating neighborhood for γ equipped with coordinates (t, z) , $z = (q_1, \dots, p_1, \dots)$, such that $x(t) = (t/T_0, 0)$, α coincides with dt on $\mathbb{R}/\mathbb{Z} \times 0$, $d\alpha|_\xi$ coincides with $\omega_0 = \sum_i dq_i \wedge dp_i$ along $\mathbb{R}/\mathbb{Z} \times 0$, and $\inf_K i_R dt > 0$. Here $B \subset \mathbb{R}^{2n-2}$ is a ball centered at the origin. This choice of coordinates induces a $d\alpha$ -symplectic trivialization of ξ along γ , which is assumed to be in the homotopy class previously chosen.

Consider a small non-degenerate perturbation α' of α on K , and $J' \in \mathcal{J}(\alpha')$ a small perturbation of J which is regular for the data (α', K, γ) as explained in 4.1.1. We denote by \mathcal{P} the set of closed α' -Reeb orbits in K homotopic to γ , and by $\mathcal{P}_0 \subset \mathcal{P}$ those which are good. Let $C_* = C_*(\alpha', K, \gamma)$ be the \mathbb{Q} -vector space freely generated by \mathcal{P}_0 graded by the Conley-Zehnder indices. Then J' can be used to define a differential d on C_* and, by Lemma 4.11, if (α', J') is sufficiently close to (α, J) the homology of (C_*, d) is the local contact homology $HC(\alpha, \gamma)$.

The natural $m : 1$ covering

$$(6.1) \quad \Pi : \tilde{K} := \mathbb{R}/m\mathbb{Z} \times \bar{B} \rightarrow K = \mathbb{R}/\mathbb{Z} \times \bar{B}$$

can be used to lift all the geometric data. $\Pi^*\alpha$ is a contact form on \tilde{K} and $\Pi^{-1}\gamma$ is an isolated $\Pi^*\alpha$ -Reeb orbit, \tilde{K} is an isolating neighborhood for $\Pi^{-1}\gamma$, $\Pi^*\alpha'$ is a small non-degenerate

perturbation of $\Pi^*\alpha$ and $(id \times \Pi)^*J' \in \mathcal{J}(\Pi^*\alpha')$ is regular for the data $(\Pi^*\alpha', \tilde{K}, \Pi^{-1}\gamma)$ and close to $(id \times \Pi)^*J$. The covering group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ of Π acts on \tilde{K} with generator

$$(6.2) \quad \sigma : \tilde{K} \rightarrow \tilde{K}, \quad (t, z) \mapsto (t + 1, z).$$

The data $(\Pi^*\alpha', (id \times \Pi)^*J')$ is invariant under this action. The lifts of closed α' -Reeb orbits homotopic to γ are precisely the closed $\Pi^*\alpha'$ -orbits which go once around the tube \tilde{K} and, consequently, they are all good. Moreover, their Conley-Zehnder indices coincide with the Conley-Zehnder indices of their projections.

Let $\tilde{\mathcal{P}}$ be the set of closed $\Pi^*\alpha'$ -Reeb orbits in \tilde{K} going once around the tube, which coincides precisely with the set of lifts of orbits in \mathcal{P} . The elements of $\tilde{\mathcal{P}}$ freely generate a \mathbb{Q} -vector space \tilde{C}_* graded by the Conley-Zehnder indices. Since J' is assumed very close to J in the C^∞ -strong topology, $(id \times \Pi)^*J'$ determines in the standard way described in Section 4 a differential \tilde{d} on \tilde{C}^* . According to Proposition 5.1, the homology of (\tilde{C}_*, \tilde{d}) coincides with the local Floer homology $HF_*(\psi^m, p)$.

Orbits in \mathcal{P} have possibly many lifts to $\tilde{\mathcal{P}}$, and the natural projection is still denoted $\Pi : \tilde{\mathcal{P}} \rightarrow \mathcal{P}$. The generator σ of the \mathbb{Z}_m -action (6.2) induces an obvious action on $\tilde{\mathcal{P}}$, and we choose a preferred lift for every element of \mathcal{P} . Our notation will be the following: if we write $\bar{\varphi}$ to denote an element in $\tilde{\mathcal{P}}$ then the chosen preferred lift is φ . Every orbit $\bar{\varphi} \in \tilde{\mathcal{P}}$ comes with a marked point $\text{pt}_{\bar{\varphi}} \in 0 \times \bar{B}$. Its multiplicity $m_{\bar{\varphi}}$ divides m and $\bar{\varphi}$ has precisely $p = m/m_{\bar{\varphi}}$ lifts which are orbits in

$$\mathcal{O}_{\bar{\varphi}} := \Pi^{-1}(\bar{\varphi}) = \{\varphi, \sigma\varphi, \dots, \sigma^{p-1}\varphi\}.$$

Note that $\sigma^{i+p}\varphi = \sigma^i\varphi$, $\forall i$. The marked point pt_{φ} is chosen in $0 \times \bar{B}$ and we set $\text{pt}_{\sigma^j\varphi} = \sigma^j(\text{pt}_{\varphi})$, so that $\Pi(\text{pt}_{\sigma^j\varphi}) = \text{pt}_{\bar{\varphi}}$, for $j = 0, \dots, p-1$. The elements of $\mathcal{O}_{\bar{\varphi}}$ are simultaneously called good/bad if $\bar{\varphi}$ is good/bad. This terminology might be troublesome since all elements of $\tilde{\mathcal{P}}$ are SFT-good (all such orbits are simple), but we will proceed without fear of ambiguity. The map $\Pi : \tilde{\mathcal{P}} \rightarrow \mathcal{P}$ induces a linear map

$$(6.3) \quad \Pi_* : \tilde{C}_* \rightarrow C_*$$

by setting $\Pi_* = \Pi$ on good generators and $\Pi_* = 0$ on bad generators. Finally set

$$(6.4) \quad \delta_{\bar{\varphi}} = \begin{cases} +1 & \text{if } \bar{\varphi} \text{ is good,} \\ -1 & \text{if } \bar{\varphi} \text{ is bad.} \end{cases}$$

6.2. Finite-energy cylinders and their lifts. Given $\eta, \zeta \in \mathcal{P}$ we denote by $\mathcal{M}(\eta, \zeta)$ the moduli spaces of finite-energy J' -holomorphic cylinders in $\mathbb{R} \times K$ with a positive and a negative puncture, asymptotic to η at its positive puncture and to ζ at its negative puncture, with asymptotic markers. Namely, an element is an equivalence class of triples (t^+, t^-, F) , where $t^\pm \in S^1$, $F = (a, f) : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times K$ is a non-constant finite-energy J' -holomorphic map with a positive puncture at $+\infty \times S^1$ where it is asymptotic to η , with a negative puncture at $-\infty \times S^1$ where it is asymptotic to ζ , and satisfying $\lim_{s \rightarrow +\infty} f(s, t^+) = \text{pt}_\eta$, $\lim_{s \rightarrow -\infty} f(s, t^-) \rightarrow \text{pt}_\zeta$. The triple (θ^+, θ^-, G) is equivalent to (t^+, t^-, F) if one finds $c, \Delta s \in \mathbb{R}$ and $\Delta t \in S^1$ satisfying $F(s, t) = \tau_c \circ G(s + \Delta s, t + \Delta t)$ and $t^\pm + \Delta t = \theta^\pm$. Differently from the notation in Section 4, here we do quotient out by the \mathbb{R} -action on the target. The equivalence class of (t^+, t^-, F) is denoted $[t^+, t^-, F]$. Moduli spaces $\mathcal{M}_0(\eta, \zeta)$ of cylinders in $\mathbb{R} \times K$ without asymptotic markers are defined as a set of equivalence classes of maps as above, where two maps F, G are equivalent if there exist $c, \Delta s \in \mathbb{R}$ and $\Delta t \in S^1$ such that

$F(s, t) = \tau_c \circ G(s + \Delta s, t + \Delta t)$. The class of such F is denoted by $[F]$, and there is a natural surjective map

$$(6.5) \quad \Delta : \mathcal{M}(\eta, \zeta) \rightarrow \mathcal{M}_0(\eta, \zeta), \quad [t^+, t^-, F] \mapsto [F].$$

Let F represent a given class $[F] \in \mathcal{M}_0(\eta, \zeta)$. Then the group $\text{Iso}(F)$ of holomorphic self-diffeomorphisms h of $\mathbb{R} \times S^1$ fixing the ends $\pm\infty \times S^1$ and satisfying $F \circ h = F$ can be identified with a subgroup of S^1 since such h must have the form $h(s, t) = (s, t + \Delta t)$. Although $\text{Iso}(F)$ depends on the representative F , its order $w[F] = \#\text{Iso}(F)$ depends only on $[F]$. The choice of F determines a subset of S^1 with m_η elements, which are possible locations of asymptotic markers at the positive puncture. The group $\text{Iso}(F)$ acts freely on this set and, consequently, $w[F]$ divides m_η . Analogously, $w[F]$ divides m_ζ . Note that \mathbb{Z}_{m_η} and \mathbb{Z}_{m_ζ} act on $\mathcal{M}(\eta, \zeta)$ by rotation of asymptotic markers, the generators are: $[t^+, t^-, F] \mapsto [t^+ + 1/m_\eta, t^-, F]$ and $[t^+, t^-, F] \mapsto [t^+, t^- + 1/m_\zeta, F]$. The set $\Delta^{-1}[F]$ has precisely $m_\eta m_\zeta / w[F]$ elements, $\forall [F] \in \mathcal{M}_0(\eta, \zeta)$.

There is a coherent system of orientations of the spaces $\mathcal{M}(\eta, \zeta)$, for all choices $\eta, \zeta \in \mathcal{P}$, compatible with glueing⁵. These are defined as in [4] even when η or ζ is bad, and determine signs $\epsilon[t^+, t^-, F] = \pm 1$ when $|\eta| - |\zeta| = 1$. One has

$$(6.6) \quad \begin{aligned} \epsilon[t^+ + 1/m_\eta, t^-, F] &= \delta_\eta \epsilon[t^+, t^-, F] \\ \epsilon[t^+, t^- + 1/m_\zeta, F] &= \delta_\zeta \epsilon[t^+, t^-, F]. \end{aligned}$$

In other words, the action of \mathbb{Z}_{m_η} in $\mathcal{M}(\eta, \zeta)$ by rotating the asymptotic marker at the positive puncture is orientation preserving/reversing when η is good/bad. The analogous statement holds for the action of \mathbb{Z}_{m_ζ} by rotations of the asymptotic marker at the negative puncture. Hence this signs descend to signs $\epsilon[F]$ on $\mathcal{M}_0(\eta, \zeta)$ only when η and ζ are good.

Moduli spaces of finite-energy $(id \times \Pi)^* J'$ -holomorphic cylinders in $\mathbb{R} \times \tilde{K}$ asymptotic to orbits in $\tilde{\mathcal{P}}$ with or without marked points are defined in the same way. However, note that all such cylinders are somewhere injective and there are no non-trivial reparametrization groups.

Choose any $\bar{\varphi}, \bar{\eta} \in \mathcal{P}$ and set $p = m/m_{\bar{\varphi}}, q = m/m_{\bar{\eta}}$. There are well-defined projections

$$(6.7) \quad \Pi_* : \mathcal{M}(\sigma^i \varphi, \sigma^j \eta) \rightarrow \mathcal{M}(\bar{\varphi}, \bar{\eta}) \quad [t^+, t^-, F] \mapsto [t^+, t^-, \Pi \circ F]$$

where $i = 0, \dots, p-1, j = 0, \dots, q-1$. For any $\zeta \in \tilde{\mathcal{P}}$ denote

$$\mathcal{M}(\mathcal{O}_{\bar{\varphi}}, \zeta) = \bigcup_{i=0}^{p-1} \mathcal{M}(\sigma^i \varphi, \zeta) \quad \text{and} \quad \mathcal{M}(\zeta, \mathcal{O}_{\bar{\eta}}) = \bigcup_{j=0}^{q-1} \mathcal{M}(\zeta, \sigma^j \eta).$$

We define the space $\mathcal{M}(\mathcal{O}_{\bar{\varphi}}, \mathcal{O}_{\bar{\eta}})$ analogously.

Any finite-energy J' -holomorphic cylinder $F = (a, f)$ representing an element in $\mathcal{M}_0(\bar{\varphi}, \bar{\eta})$ can be lifted to (possibly many) finite-energy $(id \times \Pi)^* J'$ -holomorphic cylinders, since the loops $t \mapsto f(s, t)$ go m times around the tube K and the projection (6.1) is pseudo-holomorphic with respect to J' and $(id \times \Pi)^* J'$. To be more precise, recall the forgetful map Δ (6.5) and

⁵The reader should note that, in view of Lemma 3.4, if $F = (a, f)$ is a cylinder representing an element of $\mathcal{M}(\eta, \zeta)$ for $\eta, \zeta \in \mathcal{P}$ then $\overline{f(\mathbb{R} \times S^1)} \subset \text{int}(K)$ because J' is assumed very close to some $J \in \mathcal{J}(\alpha)$. Hence, assuming regularity, such cylinders can be glued to obtain cylinders which again project into a compact subset of $\text{int}(F)$.

consider the bijection

$$(6.8) \quad \left\{ t_0^+ + \frac{k}{m_{\tilde{\varphi}}} : k = 0, \dots, \frac{m_{\tilde{\varphi}}}{w[F]} - 1 \right\} \times \left\{ t_0^- + \frac{k}{m_{\tilde{\eta}}} : k = 0, \dots, m_{\tilde{\eta}} - 1 \right\} \rightarrow \Delta^{-1}[F]$$

$$(t^+, t^-) \mapsto [t^+, t^-, F]$$

For each fixed $i \in \{0, \dots, p-1\}$, a given choice of asymptotic marker t^+ at $+\infty \times S^1$ uniquely determines a lift $\tilde{F} = (\tilde{a}, \tilde{f})$ of F to $\mathbb{R} \times \tilde{K}$ asymptotic to the orbit $\sigma^i \varphi$ at the positive puncture and satisfying $\tilde{f}(s, t^+) \rightarrow \text{pt}_{\sigma^i \varphi}$ as $s \rightarrow +\infty$. After this is done there is no control at the negative puncture: the asymptotic limit $\sigma^j \eta$ is forced on us, together with the unique location of the asymptotic marker t^- which satisfies $\tilde{f}(s, t^-) \rightarrow \text{pt}_{\sigma^j \eta}$ as $s \rightarrow -\infty$. One must have

$$\lim_{s \rightarrow +\infty} \tilde{f}\left(s, t^+ + \frac{k}{m_{\tilde{\varphi}}}\right) = \sigma^{kp}(\text{pt}_{\sigma^i \varphi}), \quad \forall k \in \mathbb{Z}.$$

Let us agree to say that $[t^+, t^-, F] \in \mathcal{M}(\tilde{\varphi}, \tilde{\eta})$ lifts to $\mathcal{M}(\sigma^i \varphi, \mathcal{O}_{\tilde{\eta}})$ when the unique lift $\tilde{F} = (\tilde{a}, \tilde{f})$ of F satisfying $\lim_{s \rightarrow +\infty} \tilde{f}(s, t^+) = \text{pt}_{\sigma^i \varphi}$ also satisfies $\lim_{s \rightarrow -\infty} \tilde{f}(s, t^-) = \text{pt}_{\sigma^j \eta}$ for the uniquely determined $\sigma^j \eta \in \mathcal{O}_{\tilde{\eta}}$ that \tilde{F} is asymptotic to at the negative puncture. Hence, out of the $m_{\tilde{\varphi}} m_{\tilde{\eta}} / w[F]$ elements of $\mathcal{M}(\tilde{\varphi}, \tilde{\eta})$ only $m_{\tilde{\varphi}} / w[F]$ of them lift to $\mathcal{M}(\sigma^i \varphi, \mathcal{O}_{\tilde{\eta}})$, for any fixed choice of i . Let us denote by \mathcal{M}^F the subset of $\mathcal{M}(\mathcal{O}_{\tilde{\varphi}}, \mathcal{O}_{\tilde{\eta}})$ consisting of the $[t^+, t^-, \tilde{F}]$ obtained lifting F as above. We concluded

$$(6.9) \quad \#(\mathcal{M}^F \cap \mathcal{M}(\sigma^i \varphi, \mathcal{O}_{\tilde{\eta}})) = \frac{m_{\tilde{\varphi}}}{w[F]}, \quad \forall i \in \{0, \dots, p-1\}.$$

Obviously, all cylinders in $\mathcal{M}(\mathcal{O}_{\tilde{\varphi}}, \mathcal{O}_{\tilde{\eta}})$ are obtained by this lifting procedure from some cylinder in $\mathcal{M}(\tilde{\varphi}, \tilde{\eta})$.

The projection Π can be used to pull-back the system of coherent orientations of moduli spaces of curves in $\mathbb{R} \times K$ to a system of coherent orientations on moduli spaces of curves in $\mathbb{R} \times \tilde{K}$:

$$\epsilon[t^+, t^-, F] = \epsilon[t^+, t^-, \Pi \circ F].$$

These are clearly compatible with glueing of curves on $\mathbb{R} \times \tilde{K}$. The generator σ of the covering group determines a bijection (again denoted by σ):

$$(6.10) \quad \sigma : \mathcal{M}(\sigma^i \varphi, \sigma^j \eta) \rightarrow \mathcal{M}(\sigma^{i+1} \varphi, \sigma^{j+1} \eta)$$

$$[t^+, t^-, F] \mapsto \left[t^+ - \frac{\delta_+(i)}{m_{\tilde{\varphi}}}, t^- - \frac{\delta_-(j)}{m_{\tilde{\eta}}}, \sigma \circ F \right]$$

where

$$(6.11) \quad \delta_+(i) = \begin{cases} 1 & \text{if } i+1 = p \\ 0 & \text{if } i+1 < p \end{cases} \quad \text{and} \quad \delta_-(j) = \begin{cases} 1 & \text{if } j+1 = q \\ 0 & \text{if } j+1 < q \end{cases}.$$

It follows that

$$(6.12) \quad \epsilon(\sigma[t^+, t^-, F]) = (\delta_{\tilde{\varphi}})^{\delta_+(i)} (\delta_{\tilde{\eta}})^{\delta_-(j)} \epsilon[t^+, t^-, F]$$

for all $[t^+, t^-, F] \in \mathcal{M}(\sigma^i \varphi, \sigma^j \eta)$.

Remark 6.2. Let $[F] \in \mathcal{M}_0(\tilde{\varphi}, \tilde{\eta})$, and let $[t^+, t^-, \tilde{F}] \in \mathcal{M}(\sigma^i \varphi, \mathcal{O}_{\tilde{\eta}})$ satisfying $\Pi \circ \tilde{F} = F$ be fixed. Then

$$\mathcal{M}(\sigma^i \varphi, \mathcal{O}_{\tilde{\eta}}) \cap \mathcal{M}^F = \{\sigma^{kp}[t^+, t^-, \tilde{F}] : k \geq 0\}, \quad \mathcal{M}^F = \{\sigma^k[t^+, t^-, \tilde{F}] : k \geq 0\}$$

for all $i \in \{0, \dots, p-1\}$.

6.3. A \mathbb{Z}_m -action on (\tilde{C}_*, \tilde{d}) by chain maps. Since $m_{\bar{\varphi}}$ is even when $\bar{\varphi}$ is bad, we can consider a linear \mathbb{Z}_m -action on \tilde{C}_* with generator E defined by

$$(6.13) \quad E(\varphi) = \sigma\varphi, \quad E(\sigma\varphi) = \sigma^2\varphi, \quad \dots, \quad E(\sigma^{p-1}\varphi) = \delta_{\bar{\varphi}}\varphi$$

on the generators of \tilde{C}_* . Our goal here is to show

Lemma 6.3. *The map E induces a \mathbb{Z}_m -action on \tilde{C}_* by chain maps.*

The lemma can be restated by saying that

$$(6.14) \quad E\tilde{d} = \tilde{d}E$$

so that we need to understand the differential \tilde{d} qualitatively. Of course, it suffices to prove (6.14) on the generators \tilde{P} .

Fix $\bar{\varphi}, \bar{\eta} \in \mathcal{P}$ satisfying $|\bar{\varphi}| - |\bar{\eta}| = 1$, and denote $p = m/m_{\bar{\varphi}}$ and $q = m/m_{\bar{\eta}}$. Each cylinder $[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})$ reveals a distinct set

$$(6.15) \quad \tilde{d}_{ij}^F \in \mathbb{Z}, \quad i \in \{0, \dots, p-1\}, \quad j \in \{0, \dots, q-1\}$$

of coefficients which, loosely speaking, is the contribution of the lifts of F to cylinders in $\mathbb{R} \times \tilde{K}$ (with all possible choices of asymptotic markers) connecting $\sigma^i\varphi$ to $\sigma^j\eta$ to the differential \tilde{d} . To be more precise, recall the set $\mathcal{M}^F \subset \mathcal{M}(\mathcal{O}_{\bar{\varphi}}, \mathcal{O}_{\bar{\eta}})$ discussed in 6.2 obtained by the lifts of F . We write

$$(6.16) \quad \mathcal{M}_{ij}^F = \mathcal{M}^F \cap \mathcal{M}(\sigma^i\varphi, \sigma^j\eta).$$

The coefficients (6.15) are defined as

$$(6.17) \quad \tilde{d}_{ij}^F = \sum_{[t^+, t^-, \tilde{F}] \in \mathcal{M}_{ij}^F} \epsilon[t^+, t^-, \tilde{F}].$$

We get the formula

$$(6.18) \quad \tilde{d}\sigma^i\varphi = \sum_{\{\bar{\eta} \in \mathcal{P}: |\bar{\eta}| = |\bar{\varphi}| - 1\}} \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \sum_{j=0}^{q-1} \tilde{d}_{ij}^F \sigma^j\eta$$

which implies

$$(6.19) \quad \langle \tilde{d}\sigma^i\varphi, \sigma^j\eta \rangle = \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \tilde{d}_{ij}^F$$

for all $(i, j) \in \{0, \dots, p-1\} \times \{0, \dots, q-1\}$.

From now on we view the indices i, j as periodic: $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_q$. Recall the functions $\delta_+ : \mathbb{Z}_p \rightarrow \{0, 1\}$, $\delta_- : \mathbb{Z}_q \rightarrow \{0, 1\}$ from (6.11). With these agreements the map E (6.13) acts on $\mathcal{O}_{\bar{\varphi}}$ and $\mathcal{O}_{\bar{\eta}}$ as

$$E\sigma^i\varphi = (\delta_{\bar{\varphi}})^{\delta_+(i)} \sigma^{i+1}\varphi, \quad E\sigma^j\eta = (\delta_{\bar{\eta}})^{\delta_-(j)} \sigma^{j+1}\eta.$$

We have

$$\langle E\tilde{d}\sigma^i\varphi, \sigma^{j+1}\eta \rangle = \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \tilde{d}_{ij}^F (\delta_{\bar{\eta}})^{\delta_-(j)}$$

and

$$\left\langle \tilde{d}E\sigma^i\varphi, \sigma^{j+1}\eta \right\rangle = \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \tilde{d}_{(i+1)(j+1)}^F(\delta_{\bar{\varphi}})^{\delta_+(i)}$$

so to prove (6.14) it suffices to show that for any $[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})$ the following identity holds

$$(6.20) \quad \tilde{d}_{ij}^F(\delta_{\bar{\eta}})^{\delta_-(j)} = \tilde{d}_{(i+1)(j+1)}^F(\delta_{\bar{\varphi}})^{\delta_+(i)}.$$

In fact, the map (6.10) maps \mathcal{M}_{ij}^F bijectively onto $\mathcal{M}_{(i+1)(j+1)}^F$. Thus

$$\begin{aligned} \tilde{d}_{(i+1)(j+1)}^F &= \sum_{[\theta^+, \theta^-, G] \in \mathcal{M}_{(i+1)(j+1)}^F} \epsilon[\theta^+, \theta^-, G] \\ &= \sum_{[t^+, t^-, \tilde{F}] \in \mathcal{M}_{ij}^F} \epsilon(\sigma[t^+, t^-, \tilde{F}]) \\ &= \sum_{[t^+, t^-, \tilde{F}] \in \mathcal{M}_{ij}^F} (\delta_{\bar{\varphi}})^{\delta_+(i)} (\delta_{\bar{\eta}})^{\delta_-(j)} \epsilon[t^+, t^-, \tilde{F}] \\ &= (\delta_{\bar{\varphi}})^{\delta_+(i)} (\delta_{\bar{\eta}})^{\delta_-(j)} \tilde{d}_{ij}^F \end{aligned}$$

which is another way of writing (6.20). In the third equality we used (6.12). This concludes the proof of Lemma 6.3.

6.4. Π_* is a chain map.

Lemma 6.4. *The map Π_* in (6.3) satisfies $\Pi_* \tilde{d} = d\Pi_*$.*

To prove the above statement we first fix arbitrary orbits $\bar{\varphi}, \bar{\eta} \in \mathcal{P}$, denote $p = m/m_{\bar{\varphi}}$, $q = m/m_{\bar{\eta}}$ and split the argument in three cases.

6.4.1. *Case 1: $\bar{\varphi}, \bar{\eta}$ are good.* Then for any $i \in \{0, \dots, p-1\}$ we have

$$\begin{aligned}
 \langle \Pi_* \tilde{d}\sigma^i \varphi, \bar{\eta} \rangle &= \left\langle \Pi_* \left(\sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \sum_{j=0}^{q-1} \tilde{d}_{ij}^F \sigma^j \eta \right), \bar{\eta} \right\rangle \\
 &= \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \sum_{j=0}^{q-1} \tilde{d}_{ij}^F \\
 &= \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \sum_{j=0}^{q-1} \sum_{[t^+, t^-, \tilde{F}] \in \mathcal{M}_{ij}^F} \epsilon[t^+, t^-, \tilde{F}] \\
 &= \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \left(\sum_{[t^+, t^-, \tilde{F}] \in \mathcal{M}^F \cap \mathcal{M}(\sigma^i \varphi, \mathcal{O}_{\bar{\eta}})} \epsilon[F] \right) \\
 &= m_{\bar{\varphi}} \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \frac{\epsilon[F]}{w[F]} \\
 &= \frac{1}{m_{\bar{\eta}}} \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \epsilon[F] \# \Delta^{-1}[F] \\
 &= \frac{1}{m_{\bar{\eta}}} \sum_{[t^+, t^-, F] \in \mathcal{M}(\bar{\varphi}, \bar{\eta})} \epsilon[t^+, t^-, F] \\
 &= \langle d\bar{\varphi}, \bar{\eta} \rangle = \langle d\Pi_* \sigma^i \varphi, \bar{\eta} \rangle.
 \end{aligned}
 \tag{6.21}$$

In the second equality we used (6.18), in the third equality we used (6.17), in the fourth equality we used that $\bar{\varphi}, \bar{\eta}$ are good, in the fifth equality we used (6.9), in the seventh equality we used that $\bar{\varphi}, \bar{\eta}$ are good and that Δ is surjective.

6.4.2. *Case 2: $\bar{\varphi}$ is bad, $\bar{\eta}$ is good.* Fix $i_0 \in \{0, \dots, p-1\}$. In this case clearly

$$\langle d\Pi_* \sigma^{i_0} \varphi, \bar{\eta} \rangle = 0$$

by the definition of Π_* (since $\bar{\varphi}$ is bad). So the work reduces to showing

$$\langle \Pi_* \tilde{d}\sigma^{i_0} \varphi, \bar{\eta} \rangle = 0. \tag{6.22}$$

For any F representing some $[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})$ we have the $p \times q$ matrix of coefficients $\tilde{d}^F = (\tilde{d}_{ij}^F)$ and, according to (6.21),

$$\langle \Pi_* \tilde{d}\sigma^i \varphi, \bar{\eta} \rangle = \sum_{[F] \in \mathcal{M}_0(\bar{\varphi}, \bar{\eta})} \sum_{j=0}^{q-1} \tilde{d}_{ij}^F \tag{6.23}$$

is the sum over all possible such matrices of the sum of the elements of the i_0 -th line. Fixing F , let $\Phi_0 = [t^+, t^-, \tilde{F}] \in \mathcal{M}_{i_0 j_0}^F$ be some reference element (as explained before, the lift $\tilde{F} = (\tilde{a}, \tilde{f})$ of F is uniquely determined by asking $\lim_{s \rightarrow +\infty} \tilde{f}(s, t^+) = \text{pt}_{\sigma^{i_0} \varphi}$, the value of j_0 is forced on us). Recalling the action σ (6.10) we have, by Remark 6.2, that $\mathcal{M}^F = \{\sigma^k \Phi_0 : k \in \mathbb{Z}\}$. From now we consider the variables i and j as periodic: $i \in \mathbb{Z}_p$, $j \in \mathbb{Z}_q$. Analogously, the indices of the matrix \tilde{d}^F will also be seen as periodic. Note that $\sigma^k \Phi_0 \in \mathcal{M}(\sigma^{i_0+k} \varphi, \sigma^{j_0+k} \eta)$,

and each element $\sigma^k \Phi_0 \in \mathcal{M}^F$ contributes with $\epsilon(\sigma^k \Phi_0) = \pm 1$ to the coefficient $d_{(i_0+k)(j_0+k)}^F$. Obviously

$$\min\{k \geq 1 : \sigma^k \Phi_0 \in \mathcal{M}^F\} = \text{lcm}(p, q).$$

However the set \mathcal{M}^F might be larger than $\{\Phi_0, \sigma\Phi_0, \dots, \sigma^{\text{lcm}(p,q)-1}\Phi_0\}$ since Φ_0 need not be equal to $\sigma^{\text{lcm}(p,q)}\Phi_0$. In fact, we have

$$\mathcal{M}^F = \{\Phi_0, \dots, \sigma^{x\text{lcm}(p,q)-1}\Phi_0\}$$

where

$$x = \min\{k \in \{1, 2, \dots\} \mid \sigma^{x\text{lcm}(p,q)}\Phi_0 = \Phi_0\}.$$

By (6.10), each walk $\{\Phi_0, \sigma\Phi_0, \dots, \sigma^{\text{lcm}(p,q)-1}\Phi_0\}$ on the matrix \tilde{d}^F corresponds to $\text{lcm}(p, q)/p$ rotations at the positive puncture, and to $\text{lcm}(p, q)/q$ rotations of the negative puncture. In view of the definition of x we have

$$(6.24) \quad [t^+, t^-, \tilde{F}] = \left[t^+ - \frac{x\text{lcm}(p, q)}{pm_{\tilde{\varphi}}}, t^- - \frac{x\text{lcm}(p, q)}{qm_{\tilde{\eta}}}, \sigma^{x\text{lcm}(p, q)} \circ \tilde{F} \right]$$

so after applying the projection (6.7) we conclude

$$[t^+, t^-, F] = \left[t^+ - \frac{x\text{lcm}(p, q)}{pm_{\tilde{\varphi}}}, t^- - \frac{x\text{lcm}(p, q)}{qm_{\tilde{\eta}}}, F \right].$$

Thus, x can be computed by

$$x = \min \left\{ k \in \{1, 2, \dots\} \text{ such that } \frac{x\text{lcm}(p, q)}{pm_{\tilde{\varphi}}} = \frac{x\text{lcm}(p, q)}{qm_{\tilde{\eta}}} \in \text{Iso}(F) \right\}$$

or, alternatively, by saying that x is the minimal positive integer for which $\exists y \in \{1, 2, \dots\}$ such that

$$\begin{cases} x \frac{\text{lcm}(p, q)}{p} = y \frac{m_{\tilde{\varphi}}}{w[F]} \\ x \frac{\text{lcm}(p, q)}{q} = y \frac{m_{\tilde{\eta}}}{w[F]} \end{cases}.$$

Substituting $y = 1$ above we would obtain

$$x = \frac{m}{\text{lcm}(p, q)w[F]}$$

since $pm_{\tilde{\varphi}} = qm_{\tilde{\eta}} = m$. Note that x given by this formula is an integer since $m/w[F]$ is a common multiple of p and q because $w[F]$ is a common divisor of $m_{\tilde{\varphi}}$ and $m_{\tilde{\eta}}$.

Observe that the path $\{\Phi_0, \dots, \sigma^{x\text{lcm}(p,q)-1}\Phi_0\}$ visits the space $\mathcal{M}(\sigma^{i_0}\varphi, \mathcal{O}_{\tilde{\eta}})$ exactly $x\text{lcm}(p, q)/p = m_{\tilde{\varphi}}/w[F]$ times, and each visit contributes with alternating signs to the i_0 -th line of \tilde{d}^F . This follows from the formula (6.10) for the action σ . Thus, in order to prove

$$(6.25) \quad \sum_{j=0}^{q-1} \tilde{d}_{i_0 j}^F = 0$$

it suffices to show that $m_{\tilde{\varphi}}/w[F]$ is even. This follows easily from the fact that the $\mathbb{Z}_{m_{\tilde{\varphi}}}$ -action on $\mathcal{M}(\tilde{\varphi}, \tilde{\eta})$ given by rotations of asymptotic markers at the positive puncture is orientation reversing. Thus (6.22) follows from (6.23) and (6.25).

This concludes *Case 2* and the proof of Lemma 6.4.

6.5. Conclusion of the proof of Proposition 6.1. Consider the average operator

$$(6.26) \quad A : \tilde{C}_* \rightarrow \tilde{C}_* \quad A = \frac{1}{m}(I + E + \cdots + E^{m-1}).$$

Since $E^m - I = 0$ we have a decomposition

$$\tilde{C}_* = \ker(E - I) \oplus \ker A = \operatorname{im} A \oplus \ker A.$$

By Lemma (6.3) we have a subcomplex $(\operatorname{im} A, \tilde{d}) \subset (\tilde{C}_*, \tilde{d})$. Let $Q : C_* \rightarrow \tilde{C}_*$ be defined on generators $\bar{\varphi} \in \mathcal{P}_0$ by

$$Q\bar{\varphi} = A(\text{some element in } \mathcal{O}_{\bar{\varphi}}).$$

Then clearly Q is injective and

$$Q\Pi_* = A.$$

Here we used that $A(\sigma^i \varphi) = 0$ whenever $\bar{\varphi}$ is a bad orbit, and that C_* is generated by the good orbits. It follows that

$$\ker \Pi_* = \ker A.$$

Here injectivity of Q was used. Thus we get that

$$\Pi_* : (\operatorname{im} A, \tilde{d}) \rightarrow (C_*, d)$$

is a linear isomorphism and a chain map. Consequently, the homology of (C_*, d) is equal to the homology of the subcomplex $(\operatorname{im} A, \tilde{d})$, and Proposition 6.1 follows immediately.

6.6. Proof of Theorem 1.1. Let $\gamma = (x, T)$ be an isolated periodic orbit with multiplicity m . Let $\Sigma \subset N$ be an embedded hypersurface transverse to γ at $\text{pt} = x(0)$, so that the local first return map $\psi : (U, \text{pt}) \rightarrow (\Sigma, \text{pt})$ is well-defined on a small neighborhood U of pt in Σ . Following [16], we say that a positive integer j is admissible for γ if $\lambda^j \neq 1$ for all eigenvalues $\lambda \neq 1$ of $d\psi^m(\text{pt})$. It follows from Proposition 6.1 and Theorem 1.1 in [16] that the total rank of $HC_*(\alpha, \gamma^j)$ is less or equal than the total rank of $HC_*(\psi^m, \text{pt})$ for every admissible j .

Now, suppose that γ is simple and every iterate of γ is isolated. In order to prove Theorem 1.1 it remains only to show that we can write the set of natural numbers as a finite union of admissible integers for some iterates of γ . This is the content of the lemma below which is extracted from [18, Lemma 2]. For the reader's convenience, we will reproduce its proof.

Lemma 6.5. *There are positive integers m_1, \dots, m_s and sequences j_k^i of natural numbers with $i \in \mathbb{N}$ and $k = 1, \dots, s$ such that the numbers $j_k^i m_k$ are mutually distinct, $\cup_{i,k} \{j_k^i m_k\} = \mathbb{N}$ and j_k^i is admissible for γ^{m_k} for every $i \in \mathbb{N}$ and $k = 1, \dots, s$.*

Proof. Assume $d\psi(\text{pt})$ has eigenvalues of the form $e^{i2\pi r}$, $r \in \mathbb{Q}$. Write the rational eigenvalues in the circle of $d\psi(\text{pt})$ in the form p/q with p and q relatively prime (to be more precise, the corresponding eigenvalue is $e^{i2\pi p/q}$), and denote by Q the set of denominators of these eigenvalues. For $\emptyset \neq A \subset Q$ let $m(A)$ denote the least common multiple of all elements in A . Choose distinct numbers m_1, \dots, m_s such that $\{m_1, \dots, m_s\} = \{m(A); \emptyset \neq A \subset Q\} \cup \{1\}$. For each $k \in \{1, \dots, s\}$ consider the set $Q_k = \{q \in Q; q \text{ does not divide } m_k\}$. We list the elements of the set $\{j \in \mathbb{N}; q \text{ does not divide } jm_k, \forall q \in Q_k\}$ in strictly increasing order $j_k^1 < j_k^2 < \dots$. Let us prove that j_k^i is admissible for γ^{m_k} . The eigenvalues in the circle for $d\psi^{m_k}(\text{pt})$ are of the form $\lambda^{m_k} \simeq m_k p/q$, for some eigenvalue $\lambda \simeq p/q$ for $d\psi(\text{pt})$ which lies in the circle. The conclusion follows since $\lambda^{m_k} \neq 1$ is equivalent to the condition that q does not divide m_k , and the condition $\lambda^{j_k^i m_k} \neq 1$ is equivalent to the condition that q does not divide $j_k^i m_k$. So it remains only to show that $\cup_{i,k} \{j_k^i m_k\} = \mathbb{N}$ but this is easy and left to the reader. \square

7. MORSE INEQUALITIES

Fix a homotopy class of augmentations $[\epsilon]$ and let α be a contact form for ξ . The action spectrum of α is given by $\Sigma(\alpha) = \{A(\gamma); \gamma \text{ is a periodic orbit of } \alpha\}$, where $A(\gamma) = \int_\gamma \alpha$ is the action of γ .

7.1. Filtered linearized contact homology. In this section we will define filtered linearized contact homology for any defining contact form for ξ . In contrast to the non-filtered homology, it depends on the choice of the contact form and the augmentation. But an augmentation is defined for the differential graded algebra associated to a *non-degenerate* contact form and we have to handle possibly *degenerate* contact forms.

In order to overcome this difficulty, we will fix a defining non-degenerate contact form α_0 for ξ , and some $J_0 \in \mathcal{J}(\alpha_0)$ assumed generic enough in order to define a differential graded algebra $(\mathcal{A}(\alpha_0), \partial_0)$ whose homology is the (full) contact homology of ξ , as explained in [9]: $\mathcal{A}(\alpha_0)$ is the supercommutative graded (by $|\cdot|$) algebra (with a unit) generated by the good closed α_0 -Reeb orbits with \mathbb{Q} coefficients ($c_1(\xi)$ is assumed to vanish), and ∂_0 is defined by the (algebraic) count of rigid punctured finite-energy spheres with one positive puncture in the symplectization $(\mathbb{R} \times N, J_0)$. We also fix a cobordism $(W_{\alpha_0}^\alpha \simeq \mathbb{R} \times N, \omega, J)$, with a positive end $([L, +\infty) \times N, d(e^t \alpha), \hat{J})$ and a negative end $((-\infty, -L] \times N, d(e^t \alpha_0), J_0)$, where $\hat{J} \in \mathcal{J}(\alpha)$, $L > 0$ and ω tames J on $[-L, L] \times N$. We briefly say that $W_{\alpha_0}^\alpha$ is a cobordism from α to α_0 . This choice can be compared to a choice of a “filling” of the contact manifold (in the fillable case the filling gives a cobordism to the empty set). Fix an augmentation ϵ_0 for α_0 with homotopy class $[\epsilon]$. For a shorter notation, we will omit the symplectic form and the almost complex structure for the cobordism although they will be tacitly assumed.

Let α^\pm be contact forms for ξ and fix a Riemannian metric on N . Given a constant $\delta > 0$ we say that a cobordism $W_{\alpha_-}^{\alpha_+} \simeq (\mathbb{R} \times N, \omega, J)$ is δ -small if ω and J satisfy the following two conditions. Firstly, there is a constant $L > 0$ and almost complex structures $J^\pm \in \mathcal{J}(\alpha^\pm)$ such that $J \in \mathcal{J}_L(J^-, J^+)$ (with $\mathcal{J}(\alpha^\pm)$ and $\mathcal{J}_L(J^-, J^+)$ as defined in 2.2.2), $\omega|_{[L, \infty) \times N} = d(e^t \alpha^+)$ and $\omega|_{(-\infty, -L] \times N} = d(e^t \alpha^-)$. Secondly,

$$\begin{aligned} d_{C^\infty([-L, L] \times N)}(J, J^+) + d_{C^\infty([-L, L] \times N)}(J, J^-) &< \delta \\ d_{C^\infty([-L, L] \times N)}(\omega, d(e^t \alpha^+)) + d_{C^\infty([-L, L] \times N)}(\omega, d(e^t \alpha^-)) &< \delta \end{aligned}$$

where $d_{C^\infty([-L, L] \times N)}$ denote (fixed) metrics for the C^∞ -topology in the corresponding spaces. Notice that given two δ -small cobordisms then their gluing is 2δ -small.

Given $-\infty \leq a < b \leq \infty$ such that $a, b \notin \Sigma(\alpha)$ and a constant $\delta > 0$, let V be a sufficiently small neighborhood of α such that $a, b \notin \Sigma(\alpha')$ for every $\alpha' \in V$ and every pair of contact forms in V can be joined by a δ -small cobordism.

Let $\alpha' \in V$ be a non-degenerate contact form and $J' \in \mathcal{J}(\alpha')$ be regular enough to get a well-defined differential graded algebra $(\mathcal{A}(\alpha'), \partial')$ whose homology is the contact homology for ξ . Let also $W_{\alpha'}^{\alpha'}$ be a δ -small cobordism from α' to α with an almost complex structure in $\mathcal{J}(\hat{J}, J')$.

Gluing $W_{\alpha'}^{\alpha'}$ and $W_{\alpha_0}^\alpha$ we obtain the cobordism $W_{\alpha_0}^{\alpha'}$. Let ϵ' be the augmentation for α' given by the pullback $\epsilon' = (\Psi_{\alpha_0}^{\alpha'})^* \epsilon_0 = \epsilon_0 \circ \Psi_{\alpha_0}^{\alpha'}$ of ϵ_0 via the chain map $\Psi_{\alpha_0}^{\alpha'} : (\mathcal{A}(\alpha'), \partial') \rightarrow (\mathcal{A}(\alpha_0), \partial_0)$ induced by $W_{\alpha_0}^{\alpha'}$. The filtered contact homology $HC^{(a, b), \epsilon'}(\alpha')$ is defined as explained in [2], since the differential respects the filtration. More precisely, let $C_*(\alpha')$ be the \mathbb{Q} -vector space freely generated by the good closed α' -Reeb orbits, graded by $|\cdot| = \mu_{CZ}(\cdot) + n - 3$,

where $\dim N = 2n - 1$. If $a, b \notin \Sigma(\alpha')$ then $C_*^a(\alpha')$ is the subspace generated by orbits with action $< a$, and $C_*^{(a,b)}(\alpha') = C_*^a(\alpha')/C_*^b(\alpha')$. The algebra $\mathcal{A}(\alpha')$ can be decomposed according to word length $\mathcal{A}(\alpha') = \mathcal{A}_0 \simeq \mathbb{Q} \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots$. If π_1 is the projection onto \mathcal{A}_1 and $S^{\epsilon'}$ is the algebra homomorphism determined by $S^{\epsilon'}(1) = 1$ and $S^{\epsilon'}(\gamma) = \gamma + \epsilon'(\gamma)$, then the linearized differential $\partial'_{\epsilon'} : C_*(\alpha') \rightarrow C_{*-1}(\alpha')$ is defined by $\pi_1 \circ S^{\epsilon'} \circ \partial'$. It satisfies $\partial'_{\epsilon'} \circ \partial'_{\epsilon'} = 0$ and $\partial'_{\epsilon'}(C_*^a(\alpha')) \subset C_*^a(\alpha')$, $\forall a \notin \Sigma(\alpha')$. Then $HC^{(a,b),\epsilon'}(\alpha')$ is the homology of the complex $(C_*^{(a,b)}(\alpha'), \partial'_{\epsilon'})$.

Proposition 7.1. *The constant δ can be chosen such that $HC^{(a,b),\epsilon'}(\alpha')$ does not depend on the choices of the δ -small cobordism from α' to α and on the non-degenerate contact form $\alpha' \in V$.*

The proof is based on the following proposition.

Proposition 7.2. *Let α' and $\tilde{\alpha}$ be non-degenerate contact forms for ξ and $\tilde{\epsilon}$ an augmentation for $\tilde{\alpha}$. Given $-\infty \leq a < b \leq \infty$ such that $a, b \notin \Sigma(\alpha') \cup \Sigma(\tilde{\alpha})$ there exists $\bar{\delta} > 0$ such that the following holds. Let $W_{\tilde{\alpha}}^{\alpha'}$ be a $\bar{\delta}$ -small cobordism from α' to $\tilde{\alpha}$ and denote by $\Psi_{\tilde{\alpha}}^{\alpha'} : (\mathcal{A}(\alpha'), \partial') \rightarrow (\mathcal{A}(\tilde{\alpha}), \tilde{\partial})$ the induced chain map. Then $HC_*^{(a,b),\tilde{\epsilon}}(\tilde{\alpha})$ and $HC_*^{(a,b),\epsilon'}(\alpha')$ are isomorphic, where $\epsilon' = (\Psi_{\tilde{\alpha}}^{\alpha'})^* \tilde{\epsilon}$.*

Proof. Since the action spectrum is a closed subset, there exists $\kappa > 0$ such that $(a - \kappa, a + \kappa) \cap (\Sigma(\alpha') \cup \Sigma(\tilde{\alpha})) = \emptyset$ and $(b - \kappa, b + \kappa) \cap (\Sigma(\alpha') \cup \Sigma(\tilde{\alpha})) = \emptyset$. Given $c > 0$ and a non-degenerate contact form α denote by $\mathcal{A}^c(\alpha)$ the subalgebra generated by the periodic orbits of α with action less than c .

Exchanging the roles of α' and $\tilde{\alpha}$, we obtain a δ -small cobordism $W_{\alpha'}^{\tilde{\alpha}}$ from $\tilde{\alpha}$ to α' . Denote by $\Phi_{\alpha'}^{\tilde{\alpha}} : (\mathcal{A}(\tilde{\alpha}), \tilde{\partial}) \rightarrow (\mathcal{A}(\alpha'), \partial')$ the induced chain map. Choose $\delta > 0$ sufficiently small such that the chain map $\Psi_{\tilde{\alpha}}^{\alpha'}$ sends $\mathcal{A}^a(\alpha')$ to $\mathcal{A}^{a+\kappa}(\tilde{\alpha})$ and $\mathcal{A}^b(\alpha')$ to $\mathcal{A}^{b+\kappa}(\tilde{\alpha})$ and $\Phi_{\alpha'}^{\tilde{\alpha}}$ sends $\mathcal{A}^a(\tilde{\alpha})$ to $\mathcal{A}^{a+\kappa}(\alpha')$ and $\mathcal{A}^b(\tilde{\alpha})$ to $\mathcal{A}^{b+\kappa}(\alpha')$.

By our choice of κ we have that Ψ and Φ preserve the subalgebras \mathcal{A}^a and \mathcal{A}^b and, perhaps after making δ smaller, the same holds for a degree $+1$ map induced by family of 2δ -small cobordisms (assumed regular) joining the gluing of $W_{\tilde{\alpha}}^{\alpha'}$ with $W_{\alpha'}^{\tilde{\alpha}}$ and the symplectization of α' that defines a chain homotopy between $\Phi \circ \Psi$ and the identity. The proof then follows from a standard argument. \square

Proof of Proposition 7.1. Let $\delta = \bar{\delta}/2$, with $\bar{\delta}$ given by the previous proposition. Let V be a neighborhood of α as before, that is, such that every pair of contact forms in V can be joined by a δ -small cobordism. We can assume that the fixed cobordism $W_{\alpha_0}^{\alpha}$ comes from a family of pairs (α_t, J_t) such that $\alpha_t = \alpha$ for $t > 1$, $\alpha_t = \alpha_0$ for $t < -1$ and α_t is non-degenerate for every t in an open and dense subset of $[-1, 1]$. Choose a value of t such that α_t is non-degenerate and contained in V . Denote the corresponding contact form by $\tilde{\alpha}$. Choosing $\tilde{\alpha}$ sufficiently close to α , we can write $W_{\alpha_0}^{\alpha}$ as a gluing of cobordisms $W_{\tilde{\alpha}}^{\alpha}$ and $W_{\alpha_0}^{\tilde{\alpha}}$, with $W_{\tilde{\alpha}}^{\alpha}$ being δ -small. The gluing of a δ -small cobordism $W_{\alpha'}^{\alpha}$ with $W_{\tilde{\alpha}}^{\alpha}$ gives a $\bar{\delta}$ -small cobordism $W_{\alpha'}^{\tilde{\alpha}}$.

Now, let $\Psi_{\alpha_0}^{\tilde{\alpha}} : (\mathcal{A}(\tilde{\alpha}), \tilde{\partial}) \rightarrow (\mathcal{A}(\alpha_0), \partial_0)$ be the chain map induced by $W_{\alpha_0}^{\tilde{\alpha}}$ and $\tilde{\epsilon} = (\Psi_{\alpha_0}^{\tilde{\alpha}})^* \epsilon_0$. By the relation $\Psi_{\alpha_0}^{\alpha'} = \Psi_{\tilde{\alpha}}^{\alpha'} \circ \Psi_{\alpha_0}^{\tilde{\alpha}}$ we have that $\epsilon' = (\Psi_{\tilde{\alpha}}^{\alpha'})^* \tilde{\epsilon}$. Consequently, by the previous proposition,

$$HC_*^{(a,b),\epsilon'}(\alpha') \simeq HC_*^{(a,b),\tilde{\epsilon}}(\tilde{\alpha}).$$

Therefore, we conclude that $HC_*^{(a,b),\epsilon'}(\alpha')$ does not depend on the choices of the small cobordism $W_{\alpha'}^{\alpha'}$ and the non-degenerate perturbation α' . \square

Thus, define $HC_*^{(a,b),\epsilon_0}(\alpha)$ as $HC_*^{(a,b),\epsilon'}(\alpha')$ for a non-degenerate contact form α' sufficiently close to α and ϵ' given by the construction above with δ given by Proposition 7.1. It follows that it does not depend on the choice of α' , but it depends on the choices of α_0 , ϵ_0 and the cobordism $W_{\alpha_0}^{\alpha_0}$. Notice however that when $a = -\infty$ and $b = \infty$ then $HC_*^{(a,b),\epsilon_0}(\alpha) \simeq HC^{[\epsilon]}(\xi)$ and consequently it does not depend on these choices at all. When $a = -\infty$ we denote $HC_*^{(-\infty,b),\epsilon_0}(\alpha)$ by $HC_*^{b,\epsilon_0}(\alpha)$ for short. The following proposition is immediate from the discussion in Section 4 and is a key ingredient in our applications.

Proposition 7.3. *Suppose that α has finitely many simple periodic orbits. Given $a \in \Sigma(\alpha)$ there exists $\delta > 0$ such that*

$$HC_*^{(a-\bar{\delta},a+\bar{\delta}),\epsilon_0}(\alpha) \simeq \oplus_{\gamma; A(\gamma)=a} HC_*(\alpha, \gamma).$$

for every $\bar{\delta} \in (0, \delta)$.

Proof. First we find $\sigma > 0$ and a C^∞ -neighborhood V of α such that $\Sigma(\alpha') \subset (\sigma, +\infty)$, $\forall \alpha' \in V$. Since $[a - \delta, a + \delta] \cap \Sigma(\alpha) = \{a\}$ if $\delta < \sigma/2$ is small enough, we can assume, possibly after making V smaller, that $a \pm \delta \notin \Sigma(\alpha')$ for any non-degenerate $\alpha' \in V$. Let us fix α' non-degenerate, and assume $J' \in \mathcal{J}(\alpha')$ regular enough so that the differential graded algebra $(\mathcal{A}(\alpha'), \partial')$ of contact homology is well-defined. Making choices α_0 and $W_{\alpha_0}^{\alpha_0}$ as explained before, we glue a very small cobordism $W_{\alpha'}^{\alpha'}$ to $W_{\alpha_0}^{\alpha_0}$ to obtain a cobordism $W_{\alpha_0}^{\alpha'}$ inducing a chain map $\Psi : (\mathcal{A}(\alpha'), \partial') \rightarrow (\mathcal{A}(\alpha_0), \partial_0)$ and we use $\Psi^* \epsilon_0$ to linearize ∂' and obtain the differential of linearized contact homology $HC_*^{(a-\delta, a+\delta), \Psi^* \epsilon_0}(\alpha')$. The proposition follows from the observation that this linearized differential is defined only by counting spheres with one negative puncture, since the presence of an extra negative punctures would drop the action by at least $\sigma > 2\delta$, giving closed Reeb orbits out of the action interval $(a - \delta, a + \delta)$. In other words, the linearization $\Psi^* \epsilon_0$ plays no role, the homology $HC_*^{(a-\delta, a+\delta), \Psi^* \epsilon_0}(\alpha')$ is cylindrical in essence. Now an easy compactness argument using the results from the Appendix will show that, after taking $\delta > 0$ smaller, we can assume that the cylinders which define the linearized differential must be connecting closed α' -Reeb orbits inside small (isolating) tubular neighborhoods of the closed α -Reeb orbits with action a . \square

7.2. Morse inequalities. Suppose now that α has finitely many simple periodic orbits $\gamma_1, \gamma_2, \dots, \gamma_r$. In particular, its action spectrum is a discrete subset. Let $b_i^{[\epsilon]} = \dim HC_i^{[\epsilon]}(\xi)$ denote the Betti numbers and

$$c_i = \sum_{k=1}^r \sum_j \dim HC_i(\alpha, \gamma_k^j)$$

the Morse type numbers. The main result in this section provides versions of weak and strong Morse inequalities for contact homology suitable for our applications.

Proposition 7.4. *Under the assumption that α has finitely many simple periodic orbits, $b_i^{[\epsilon]}$ and c_i are finite for every $i \geq 2n - 3$ and satisfy the inequalities*

$$(7.1) \quad b_i^{[\epsilon]} \leq c_i$$

and

$$(7.2) \quad b_i^{[\epsilon]} - b_{i-1}^{[\epsilon]} + \dots \pm b_{2n-3}^{[\epsilon]} - C \leq c_i - c_{i-1} + \dots \pm c_{2n-3},$$

for every $i \geq 2n - 3$, where $C \geq 0$ is a constant that does not depend on i .

Remark 7.5. The analogous result holds for $i \leq -3$. This condition ensures that only periodic orbits with negative mean index can contribute to $b_i^{[\epsilon]}$ and c_i .

Proof. Let γ be a periodic orbit that split into non-degenerate orbits ψ_1, \dots, ψ_l under a C^∞ -small, non-degenerate perturbation of α . Then it is well known that

$$(7.3) \quad |\mu_{\text{CZ}}(\psi_i) - \Delta(\gamma)| \leq n - 1$$

for every $i = 1, \dots, l$, see [16, page 331]. Thus we have that only periodic orbits with positive mean index can contribute to c_i and $b_i^{[\epsilon]}$ for $i \geq 2n - 3$. Since there are finitely many simple closed orbits, we conclude that both $b_i^{[\epsilon]}$ and c_i are finite for every $i \geq 2n - 3$.

As in Section 7.1, fix a non-degenerate contact form α_0 for ξ , a regular cobordism $W_\alpha^{\alpha_0}$ from α to α_0 and an augmentation ϵ_0 for α_0 with homotopy class $[\epsilon]$. Given $a \notin \Sigma(\alpha)$, let b_i^{a, ϵ_0} be the rank of $HC_i^{a, \epsilon_0}(\alpha)$ and

$$c_i^a = \sum_{k=1}^r \sum_{j; A(\gamma_k^j) < a} \dim HC_i(\alpha, \gamma_k^j)$$

the relative Morse type numbers.

Lemma 7.6. Given $i \geq 2n - 3$ there exists $a \notin \Sigma(\alpha)$ such that $b_l^{a, \epsilon_0} = b_l^{[\epsilon]}$ and $c_l^a = c_l$ for every $2n - 3 \leq l \leq i$.

Proof. To prove the first assertion, notice that by the long exact sequence

$$\dots \rightarrow HC_{*+1}^{(a, \infty), \epsilon_0}(\alpha) \rightarrow HC_*^{a, \epsilon_0}(\alpha) \rightarrow HC_*^{[\epsilon]}(\xi) \rightarrow HC_*^{(a, \infty), \epsilon_0}(\alpha) \rightarrow \dots$$

it is enough to show that there exists $a > 0$ such that

$$(7.4) \quad HC_{l+1}^{(a, \infty), \epsilon_0}(\alpha) = HC_l^{(a, \infty), \epsilon_0}(\alpha) = 0$$

for every $2n - 3 \leq l \leq i$. Given $a > 0$ such that $a \notin \Sigma(\alpha)$ let $a_j \rightarrow \infty$, $j \in \mathbb{N}^0$, be an increasing sequence of real numbers such that $a_0 = a$ and $a_j \notin \Sigma(\alpha)$ for every j . The inclusions $CC_l^{(a, a_j)}(\alpha) \hookrightarrow CC_l^{(a, a_{j+1})}(\alpha)$ induce applications in the homology that make $HC_l^{(a, a_j), \epsilon_0}(\alpha)$ a direct system. In particular, we have its direct limit $\lim_{j \rightarrow \infty} HC_l^{(a, a_j), \epsilon_0}(\alpha)$. It turns out that there exists an isomorphism

$$(7.5) \quad HC_l^{(a, \infty), \epsilon_0}(\alpha) \simeq \lim_{j \rightarrow \infty} HC_l^{(a, a_j), \epsilon_0}(\alpha),$$

for every l , see [8]. Let $a \notin \Sigma(\alpha)$ be a positive real number such that every periodic orbit with positive mean index and action greater than a has mean index bigger than $i + 3$. The existence of such a follows from the fact that there are finitely many simple periodic orbits. By inequality (7.3), given $j > 0$ there exists a non-degenerate small perturbation $\tilde{\alpha}$ of α with no closed orbits having action in (a, a_j) and degree in $[2n - 3, i + 1]$. Consequently, $HC_l^{(a, a_j), \epsilon_0}(\alpha) \simeq HC_l^{(a, a_j), \epsilon'}(\alpha') = 0$ for every $l \in [2n - 3, i + 1]$, where ϵ' is the augmentation for α' obtained from the discussion in Section 7.1. Equation (7.4) then follows from the isomorphism (7.5). \square

Now, let $i \geq 2n - 3$ and a as in the previous lemma. Let m be an integer (that depends on a) such that $m < 2n - 3$ and every periodic orbit with mean index in $[m - 2n + 4, m + 2]$ has action bigger than a (the existence of m follows from the assumption that there are finitely many simple periodic orbits). By (7.3), we have that $HC_m^{(a', a''), \epsilon_0}(\alpha) = 0$ for every $0 \leq a' < a'' \leq a$. Then it follows from the long exact sequence for filtered contact homology that the function

$$\chi_l(a', a'') := \dim HC_l^{(a', a''), \epsilon_0}(\alpha) - \dim HC_{l-1}^{(a', a''), \epsilon_0}(\alpha) + \dots \pm \dim HC_m^{(a', a''), \epsilon_0}(\alpha)$$

is subadditive, that is, given $0 < a' < a'' < a''' \leq a$ then $\chi_l(a', a''') \leq \chi_l(a', a'') + \chi_l(a'', a''')$ for every $l \geq m$. This implies in a standard way the strong Morse inequality

$$b_l^{a, \epsilon_0} - b_{l-1}^{a, \epsilon_0} + \dots \pm b_m^{a, \epsilon_0} \leq c_l^a - c_{l-1}^a + \dots \pm c_m^a.$$

Notice here that b_l^{a, ϵ_0} and c_l^a are finite for every $l \in \mathbb{Z}$. Applying this inequality for $l = i$ and $l = i + 1$ we arrive at

$$b_i^{[\epsilon]} = b_i^{a, \epsilon_0} \leq c_i^a = c_i$$

proving (7.1). Inequality (7.2) does not follow directly from the previous discussion due to the fact that m depends on a . To circumvent this problem, the idea is to truncate the action filtration from below in a suitable way. The price that we have to pay is to add a correction term in the inequalities which in turn does not depend on a .

Let $\tilde{a} \notin \Sigma(\alpha)$ be such that $\tilde{a} < a$ and every periodic orbit with positive mean index and action bigger than \tilde{a} has mean index greater than $2n - 1$ (again the existence of \tilde{a} follows from the assumption that there are finitely many simple periodic orbits). By (7.3) we conclude that $HC_{2n-3}^{(a', a''), \epsilon_0}(\alpha) = 0$ for every $\tilde{a} \leq a' < a''$. We have then that

$$\chi_l(a', a'') := \dim HC_l^{(a', a''), \epsilon_0}(\alpha) - \dim HC_{l-1}^{(a', a''), \epsilon_0}(\alpha) + \dots \pm \dim HC_{2n-3}^{(a', a''), \epsilon_0}(\alpha)$$

is subadditive, that is, given $\tilde{a} < a' < a'' < a''' \leq a$ then $\chi_l(a', a''') \leq \chi_l(a', a'') + \chi_l(a'', a''')$ for every $l \geq 2n - 3$. This implies the inequality

$$(7.6) \quad b_l^{(\tilde{a}, a), \epsilon_0} - b_{l-1}^{(\tilde{a}, a), \epsilon_0} + \dots \pm b_{2n-3}^{(\tilde{a}, a), \epsilon_0} \leq c_l^{(\tilde{a}, a)} - c_{l-1}^{(\tilde{a}, a)} + \dots \pm c_{2n-3}^{(\tilde{a}, a)},$$

where $b_l^{(\tilde{a}, a), \epsilon_0}$ denotes the rank of $HC_l^{(\tilde{a}, a), \epsilon_0}(\alpha)$ and

$$c_l^{(\tilde{a}, a)} := \sum_{k=1}^r \sum_{j; \tilde{a} \leq A(\gamma_k^j) \leq a} \dim HC_l(\alpha, \gamma_k^j).$$

Let $\Delta_{\tilde{a}} := \max_{A(\gamma) \leq \tilde{a}} \Delta(\gamma)$ and take $i > \Delta_{\tilde{a}} + 2n$. A inspection in the proof of Lemma 7.6 (a is such that every periodic orbit with positive mean index and action greater than a has mean index bigger than $i + 3$) shows that every periodic orbit with mean index in $(\Delta_{\tilde{a}}, i + 3)$ has action in (\tilde{a}, a) . But if $|\gamma| \in (\Delta_{\tilde{a}} + 2n - 4, i + 1)$ then $\Delta(\gamma) \in (\Delta_{\tilde{a}}, i + 3)$. Hence $HC_l^{(\tilde{a}, a), \epsilon_0}(\alpha) \simeq HC_l^{[\epsilon]}(\alpha)$ for every integer $l \in (\Delta_{\tilde{a}} + 2n - 4, i + 1)$. Since $\Delta_{\tilde{a}}$ does not depend on i , we clearly conclude from (7.6) and the previous isomorphism that there exists $C \geq 0$ that does not depend on i such that

$$b_i^{[\epsilon]} - b_{i-1}^{[\epsilon]} + \dots \pm b_{2n-3}^{[\epsilon]} - C \leq c_i - c_{i-1} + \dots \pm c_{2n-3}$$

for every $i \geq 2n - 3$. □

Remark 7.7. One easily conclude from the proof of Proposition 7.4 the Morse inequalities for filtered contact homology, that is,

$$(7.7) \quad b_i^{a, \epsilon_0} \leq c_i^a$$

and

$$(7.8) \quad b_i^{a, \epsilon_0} - b_{i-1}^{a, \epsilon_0} + \dots \pm b_{2n-3}^{a, \epsilon_0} - C \leq c_i^a - c_{i-1}^a + \dots \pm c_{2n-3}^a,$$

for every $a \notin \Sigma(\alpha)$ and a constant $C \geq 0$ that does not depend on i , a and ϵ_0 .

8. PROOF OF THE APPLICATIONS

8.1. Proof of Theorem 1.2. An important ingredient in the proof of our applications is the following lemma that gives uniform bounds for the Morse type numbers of periodic orbits with mean index different from zero and follows easily from Theorem 1.1.

Lemma 8.1. *Let γ be an isolated periodic orbit of the Reeb flow of α with mean index different from zero such that γ^j is isolated for every $j \in \mathbb{N}$. There exists a constant $B > 0$ such that*

$$\sum_j \dim HC_i(\alpha, \gamma^j) < B$$

for every $i \in \mathbb{Z}$.

Proof. Since γ^j is isolated for every $j \in \mathbb{N}$, we conclude from Theorem 1.1 that there exists a constant $C > 0$ such that

$$\dim HC_i(\alpha, \gamma^j) < C$$

for every $i \in \mathbb{Z}$ and $j \in \mathbb{N}$. By (7.3) we have that $HC_*(\alpha, \gamma^j)$ is supported in the interval $[j\Delta(\gamma) - 2, j\Delta(\gamma) + 2n - 4]$, that is, $HC_i(\alpha, \gamma^j) = 0$ if $i \notin [j\Delta(\gamma) - 2, j\Delta(\gamma) + 2n - 4]$. Now the result follows easily. \square

Proof of Theorem 1.2. We will prove the result in the case that there exists a positive sequence of integers $l_i \rightarrow \infty$ such that $b_{l_i}^{[\epsilon]}(\xi) \rightarrow \infty$ since the negative case is analogous. Suppose that there exists a contact form for ξ with finitely many simple closed orbits. By inequality (7.3) only periodic orbits with positive mean index can contribute to c_i for $i \geq 2n - 3$. By Lemma 8.1 there exists a constant $B > 0$ such that $c_i < B$ for every $i \geq 2n - 3$. Hence by our assumption and inequality (7.1) we obtain a contradiction. \square

8.2. Invariance of the growth rate. We will reproduce the argument of Seidel in [30, Section 4a] that shows the invariance of the growth rate for symplectic cohomology under Liouville isomorphisms. However, the argument has to be adapted to our context, where we have to deal with augmentations. Let α_0 and α_1 be two non-degenerate contact forms for ξ and $W_{\alpha_1}^{\alpha_0}$ a cobordism from α_0 to α_1 . This cobordism induces a chain map $\Psi : (\mathcal{A}(\alpha_0), \partial_0) \rightarrow (\mathcal{A}(\alpha_1), \partial_1)$. An augmentation ϵ for $(\mathcal{A}(\alpha_1), \partial_1)$ yields an augmentation $\Psi^*\epsilon$ for $(\mathcal{A}(\alpha_0), \partial_0)$ and Ψ induces an isomorphism $\tilde{\Psi}_\epsilon : HC^{\Psi^*\epsilon}(\alpha_0) \rightarrow HC^\epsilon(\alpha_1)$. As in Section 7.1, given $a > 0$ and a contact form α let $\mathcal{A}^a(\alpha)$ be the subalgebra generated by the periodic orbits of α with action less than a . It turns out that there exists a constant $D_1 > 0$ such that Ψ sends $\mathcal{A}^a(\alpha_0)$ to $\mathcal{A}^{D_1 a}(\alpha_1)$ for every $a > 0$.

Exchanging the roles of α_0 and α_1 we obtain a chain homomorphism $\Phi : (\mathcal{A}(\alpha_1), \partial_1) \rightarrow (\mathcal{A}(\alpha_0), \partial_0)$ that sends $\mathcal{A}^a(\alpha_1)$ to $\mathcal{A}^{D_2 a}(\alpha_0)$ for every $a > 0$, where $D_2 > 0$ is a constant. The augmentations $\Phi^*\Psi^*\epsilon$ and ϵ are homotopic, that is, there exists a derivation K such

that $\epsilon = \Phi^* \Psi^* \epsilon \circ e^{\partial_1 \circ K + K \circ \partial_1}$. One can check that there exists $D_3 > 0$ such that the chain homomorphism $e^{\partial_1 \circ K + K \circ \partial_1}$ sends $\mathcal{A}^a(\alpha_1)$ to $\mathcal{A}^{D_3 a}(\alpha_1)$ for every $a > 0$.

Now, let $D = \max_i D_i$. It turns out that the induced maps in the homology fit into the ladder-shaped commutative diagram

$$(8.1) \quad \begin{array}{ccc} & \cdots & \cdots \\ & \nwarrow & \nwarrow \\ & HC_*^{D^8 a, \Phi^* \Psi^* \epsilon}(\alpha_1) & \\ & \nwarrow & \nwarrow \\ HC_*^{D^6 a, \Psi^* \epsilon}(\alpha_0) & \xrightarrow{\quad} & HC_*^{D^7 a, \epsilon}(\alpha_1) \\ & \nwarrow & \nwarrow \\ & HC_*^{D^5 a, \Phi^* \Psi^* \epsilon}(\alpha_1) & \\ & \nwarrow & \nwarrow \\ HC_*^{D^3 a, \Psi^* \epsilon}(\alpha_0) & \xrightarrow{\quad} & HC_*^{D^4 a, \epsilon}(\alpha_1) \\ & \nwarrow & \nwarrow \\ & HC_*^{D^2 a, \Phi^* \Psi^* \epsilon}(\alpha_1) & \\ & \nwarrow & \nwarrow \\ HC_*^{a, \Psi^* \epsilon}(\alpha_0) & \xrightarrow{\quad} & HC_*^{Da, \epsilon}(\alpha_1) \end{array}$$

where the maps in the vertical arrows are those induced by the inclusion.

Now, suppose that $HC_*^{\Psi^* \epsilon}(\alpha_0) \simeq HC_*^{\epsilon}(\alpha_1)$ is infinite-dimensional, since, otherwise, we would have $\Gamma^{\Psi^* \epsilon}(\alpha_0) = \Gamma^{\epsilon}(\alpha_1) = 0$. Then,

$$\begin{aligned} \Gamma^{\Psi^* \epsilon}(\alpha_0)^{-1} &= \liminf_{a \rightarrow \infty} \frac{\log a}{\log r(\Psi^* \epsilon, \alpha_0, a)} = \liminf_{a \rightarrow \infty} \frac{\log Da}{\log r(\Psi^* \epsilon, \alpha_0, a)} \\ &\geq \liminf_{a \rightarrow \infty} \frac{\log Da}{\log r(\epsilon, \alpha_1, a)} = \Gamma^{\epsilon}(\alpha_1)^{-1}, \end{aligned}$$

where $r(\epsilon, \alpha, a)$ is the rank of $\iota(HC_*^{a, \epsilon}(\alpha))$. Inverting the roles of α_0 and α_1 we conclude that the set $\{\Gamma^{\epsilon}(\alpha); \epsilon \text{ is an augmentation for } \alpha\}$ is an invariant of the contact structure.

8.3. Proof of Theorem 1.4. Suppose that there is a contact form α for ξ with finitely many simple periodic orbits $\gamma_1, \dots, \gamma_r$. As in Section 7.4 fix a non-degenerate contact form α_0 for ξ , an augmentation ϵ_0 for α_0 and a cobordism joining α_0 and α . Choose ϵ_0 such that $\Gamma^{\epsilon_0}(\alpha_0) > 1$. Then, arguing as in Section 8.2, we conclude that

$$(8.2) \quad \limsup_{a \rightarrow \infty} \frac{1}{\log a} \log \dim \iota(HC^{a, \epsilon_0}(\alpha)) > 1.$$

By Proposition 7.4 and Remark 7.7,

$$\dim HC_*^{a, \epsilon_0}(\alpha) \leq \sum_{i=1}^r \sum_{j; A(\gamma_i^j) < a} \dim HC_*(\alpha, \gamma_i^j)$$

for every $a \notin \Sigma(\alpha)$. By Theorem 1.1 there are constants B_i such that

$$\dim HC_*(\alpha, \gamma_i^j) < B_i$$

for every $1 \leq i \leq r$ and $j \in \mathbb{N}$. Consequently,

$$\dim HC_*^{a, \epsilon_0}(\alpha) \leq \frac{r \max_{1 \leq i \leq r} B_i}{\min_{1 \leq i \leq r} A(\gamma_i)} a,$$

contradicting (8.2). \square

8.4. Resonance relations.

Proof of Theorem 1.5. We will prove the result for the positive mean Euler characteristic since the negative case is analogous. Let α be a contact form with finitely many simple periodic orbits and denote by $\gamma_1, \gamma_2, \dots, \gamma_r$ those with positive mean index. We claim that

$$\chi_+^{[\epsilon]}(\xi) = \sum_{k=1}^r \lim_{m \rightarrow \infty} \frac{1}{m} \sum_j \sum_{i=2n-3}^m (-1)^i \dim HC_i(\alpha, \gamma_k^j).$$

Indeed, let B_m and C_m be the left and right sides of inequality (7.2) respectively for $i = m$. Using (7.2) for $m+1$ and m and Lemma 8.1 we conclude that there exist constants B and C such that

$$|B_m - C_m| \leq C + c_{m+1} - b_{m+1}^{[\epsilon]} \leq C + B$$

for every $m \geq 2n-3$. Consequently,

$$\begin{aligned} \chi_+^{[\epsilon]}(\xi) &= \lim_{m \rightarrow \infty} \frac{(-1)^m}{m} B_m = \lim_{m \rightarrow \infty} \frac{(-1)^m}{m} C_m \\ &= \sum_{k=1}^r \lim_{m \rightarrow \infty} \frac{1}{m} \sum_j \sum_{i=2n-3}^m (-1)^i \dim HC_i(\alpha, \gamma_k^j) \end{aligned}$$

as claimed. Now, notice that for any periodic orbit γ with positive mean index we have

$$\begin{aligned} \sum_{j=1}^m \chi_+(\alpha, \gamma^j) - \left(\frac{2(n-1)}{\Delta(\gamma)} + 1 \right) B &\leq \sum_{j=1}^m \chi_+(\alpha, \gamma^j) - \sum_{j=m+1}^{\lfloor m+2(n-1)/\Delta(\gamma) \rfloor + 1} \sum_i \dim HC_i(\alpha, \gamma^j) \\ &\leq \sum_j \sum_{i=0}^{\lfloor m\Delta(\gamma) + 2n-4 \rfloor} (-1)^i \dim HC_i(\alpha, \gamma^j) \\ &\leq \sum_{j=1}^m \chi_+(\alpha, \gamma^j) + \sum_{j=m+1}^{\lfloor m+2(n-1)/\Delta(\gamma) \rfloor + 1} \sum_i \dim HC_i(\alpha, \gamma^j) \\ &\leq \sum_{j=1}^m \chi_+(\alpha, \gamma^j) + \left(\frac{2(n-1)}{\Delta(\gamma)} + 1 \right) B, \end{aligned}$$

for every $m \in \mathbb{N}$, where $\lfloor x \rfloor = \max\{m \in \mathbb{Z}; m \leq x\}$. The first and fourth inequalities follow from Lemma 8.1. The second and third inequalities hold because if $j > m+2(n-1)/\Delta(\gamma)$ then $j\Delta(\gamma) - 2 > m\Delta(\gamma) + 2n - 4$ and by (7.3) the local contact homology satisfies $HC_i(\alpha, \gamma^j) = 0$ if $i \notin [j\Delta(\gamma) - 2, j\Delta(\gamma) + 2n - 4]$ for every j .

By the previous inequalities we arrive at

$$\begin{aligned}
\chi_+^{[\epsilon]}(\xi) &= \sum_{k=1}^r \lim_{m \rightarrow \infty} \frac{1}{m} \sum_j \sum_{i=2n-3}^m (-1)^i \dim HC_i(\alpha, \gamma_k^j) \\
&= \sum_{k=1}^r \lim_{m \rightarrow \infty} \frac{1}{m} \sum_j \sum_{i=0}^m (-1)^i \dim HC_i(\alpha, \gamma_k^j) \\
&= \sum_{k=1}^r \lim_{m \rightarrow \infty} \frac{1}{[m\Delta(\gamma_k) + 2n - 4]} \sum_{j=1}^m \chi_+(\alpha, \gamma_k^j) \\
&= \sum_{k=1}^r \lim_{m \rightarrow \infty} \frac{m}{[m\Delta(\gamma_k) + 2n - 4]} \left(\frac{1}{m} \sum_{j=1}^m \chi_+(\alpha, \gamma_k^j) \right) \\
&= \sum_{k=1}^r \lim_{m \rightarrow \infty} \frac{\frac{1}{m} \sum_{j=1}^m \chi_+(\alpha, \gamma_k^j)}{\Delta(\gamma_k)} \\
&= \sum_{k=1}^r \frac{\hat{\chi}_+(\alpha, \gamma_k)}{\Delta(\gamma_k)}.
\end{aligned}$$

□

Proof of Corollary 1.6. When γ^j is non-degenerate for every j it is easy to see that

$$\chi_{\pm}(\gamma) = \begin{cases} (-1)^{|\gamma|} & \text{if } \gamma \text{ is good} \\ (-1)^{|\gamma|}/2 & \text{if } \gamma \text{ is bad} \end{cases}.$$

□

8.5. Non-hyperbolic periodic orbits. The proofs of Theorems 1.7 and 1.8 follow from the two theorems below.

Theorem 8.2. *Suppose that there are finitely many simple closed orbits, all of them hyperbolic. If $\dim HC_{n-3}(\xi) < \infty$ then there is no closed orbit with Conley-Zehnder index equal to zero.*

Proof. Arguing indirectly, let γ be a closed orbit of index zero. It is well known that the index of a hyperbolic periodic orbit ψ satisfies

$$(8.3) \quad \mu_{CZ}(\psi^k) = k\mu_{CZ}(\psi)$$

for every $k \in \mathbb{N}$. In particular, we conclude that $\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma) = 0$ for every $k \in \mathbb{N}$. By equality (8.3) and our assumption that there are finitely many simple closed orbits, we also conclude that there are finitely many periodic orbits of index -1 and 1 . Hence, since the differential decreases the action, we have that a chain generated by orbits of index zero and action big enough cannot be exact. In particular, there exists $k_0 > 0$ such that every chain $\sum_i a_i \gamma^{k_i}$ cannot be exact as long as $k_i > k_0$ for every i .

Let ψ_1, \dots, ψ_N be the periodic orbits of index -1 . The set of chains of degree $n - 4$ can be naturally identified with \mathbb{Q}^N :

$$\sum_{i=1}^N a_i \psi_i \longleftrightarrow (a_1, \dots, a_N) \in \mathbb{Q}^N.$$

Therefore, given $k \in \mathbb{N}$ we can identify $\partial\gamma^k$ with a vector $v_k := (a_1^k, \dots, a_N^k)$ determined by the relation $\partial\gamma^k = \sum_{i=1}^N a_i^k \psi_i$. Consequently, given $k_0 < k_1 < \dots < k_{N+1}$ we have that $v_{k_1}, \dots, v_{k_{N+1}}$ must be linearly dependent, that is, there exist rational numbers p_1, \dots, p_{N+1} such that

$$p_1 v_{k_1} + \dots + p_{N+1} v_{k_{N+1}} = 0.$$

Thus the chain $p_1 \gamma^{k_1} + \dots + p_{N+1} \gamma^{k_{N+1}}$ is closed and not exact. Since one can take k_0 arbitrarily large, we conclude that $\dim HC_{n-3}^{[\epsilon]}(\xi) = \infty$. \square

Proof of Theorem 1.8. Arguing indirectly, suppose that every periodic orbit is hyperbolic. Since every periodic orbit is hyperbolic and therefore non-degenerate, the hypothesis that $\dim HC_{n-3}^{[\epsilon]}(\xi) > 0$ implies that there exists a closed orbit γ of index zero. This contradicts Theorem 8.2. \square

Theorem 8.3. *Suppose that there is no periodic orbit with Conley-Zehnder index equal to zero and that there exists a positive integer C such that*

$$(-1)^n \sum_{i=n-3}^{mC+n-3} (-1)^i b_i^{[\epsilon]}(\xi) < (-1)^n mC \chi_+^{[\epsilon]}(\xi)$$

for every $m \in \mathbb{N}$. Suppose that there are finitely many simple periodic orbits with positive mean index. Then there is a non-hyperbolic closed orbit.

Proof. Arguing indirectly, suppose that every closed orbit is hyperbolic. In particular, every periodic orbit is non-degenerate. Firstly, let us show that there is at least one periodic orbit with positive mean index. Indeed, if there is no such orbit we would have that $\chi_+^{[\epsilon]}(\xi) = 0$ and, by the equality (8.3), $b_i^{[\epsilon]} = 0$ for every $i > n - 3$. Moreover, the hypothesis that there is no periodic orbit of index zero implies that $b_{n-3}^{[\epsilon]} = 0$. This contradicts the hypothesis that $(-1)^n \sum_{i=n-3}^{mC+n-3} (-1)^i b_i^{[\epsilon]}(\xi) < (-1)^n mC \chi_+^{[\epsilon]}(\xi)$.

So let $\gamma_1, \dots, \gamma_r$ be the simple periodic orbits with positive mean index. By (8.3) we have that $\Delta(\gamma_k) = \mu_{\text{CZ}}(\gamma_k)$ is an integer for every $1 \leq k \leq r$. Let $C' = \text{lcm}\{C, 2 \prod_{k=1}^r \Delta(\gamma_k)\}$ and $c_i(\gamma_k) = \sum_j \dim HC_i(\alpha, \gamma_k^j)$. By the relation (8.3) we conclude that

$$\sum_{i=n-3}^{m\Delta(\gamma_k)+n-3} (-1)^i c_i(\gamma_k) = \epsilon_k (-1)^{\Delta(\gamma_k)+n-3} m$$

for every $1 \leq k \leq r$ and $m \in \mathbb{N}$, where $\epsilon_k = 1$ if γ_k^2 is good and $\epsilon_k = 1/2$ if γ_k^2 is bad. From this, we arrive at

$$\sum_{i=n-3}^{mC'+n-3} (-1)^i c_i(\gamma_k) = \epsilon_k mC' \frac{(-1)^{\Delta(\gamma_k)+n-3}}{\Delta(\gamma)},$$

since $mC' = (mC'/\Delta(\gamma))\Delta(\gamma)$ and $mC'/\Delta(\gamma) \in \mathbb{N}$. Hence, by Corollary 1.6,

$$\sum_{k=1}^r \sum_{i=n-3}^{mC'+n-3} (-1)^i c_i(\gamma_k) = mC' \chi_+^{[\epsilon]}(\xi).$$

Now, we need the following special version of the strong Morse inequality.

Lemma 8.4. *Suppose that there is no periodic orbit of index zero and every periodic orbit is non-degenerate. Then*

$$b_i^{[\epsilon]} - b_{i-1}^{[\epsilon]} + \dots \pm b_{n-3}^{[\epsilon]} \leq c_i - c_{i-1} + \dots \pm c_{n-3}$$

for every $i \geq n-3$.

Proof. Fix an augmentation ϵ_0 for α with homotopy class $[\epsilon]$. Since every periodic is non-degenerate and there is no periodic orbit of index zero we have that $HC_{n-3}^{(a,b),\epsilon_0}(\alpha) = 0$ for every $-\infty \leq a < b \leq \infty$. In particular, $b_{n-3}^{[\epsilon]} = c_{n-3} = 0$. Moreover, by the long exact sequence for filtered contact homology, the function

$$\chi_l(a, b) = \dim HC_l^{(a,b),\epsilon_0}(\alpha) - \dim HC_{l-1}^{(a,b),\epsilon_0}(\alpha) + \dots \pm \dim HC_{n-2}^{(a,b),\epsilon_0}(\alpha)$$

is subadditive for every $l \geq n-2$. This implies the inequality

$$b_i^{[\epsilon]} - b_{i-1}^{[\epsilon]} + \dots \pm b_{n-3}^{[\epsilon]} = b_i^{[\epsilon]} - b_{i-1}^{[\epsilon]} + \dots \mp b_{n-2}^{[\epsilon]} \leq c_i - c_{i-1} + \dots \mp c_{n-2} = c_i - c_{i-1} + \dots \pm c_{n-3}$$

for every $i \geq n-3$, as desired. \square

By the previous lemma, we have that

$$(-1)^{n-1} \sum_{k=1}^r \sum_{i=n-3}^{mC'+n-3} (-1)^i c_i(\gamma_k) \geq (-1)^{n-1} \sum_{i=n-3}^{mC'+n-3} (-1)^i b_i^{[\epsilon]}(\xi),$$

where the term $(-1)^{n-1}$ comes from the fact that the term with exponent $mC' + n - 3$ must have positive sign (notice that C' is even). Consequently,

$$\begin{aligned} (-1)^{n-1} mC' \chi_+^{[\epsilon]}(\xi) &= (-1)^{n-1} \sum_{k=1}^r \sum_{i=n-3}^{mC'+n-3} (-1)^i c_i(\gamma_k) \\ &\geq (-1)^{n-1} \sum_{i=n-3}^{mC'+n-3} (-1)^i b_i^{[\epsilon]}(\xi) \\ &> (-1)^{n-1} mC' \chi_+^{[\epsilon]}(\xi), \end{aligned}$$

contradiction. \square

Proof of Theorem 1.7. The result for the positive Euler characteristic is immediate by Theorems 8.2 and 8.3. The argument for the negative Euler characteristic is analogous. \square

APPENDIX A. FINITE-ENERGY CURVES AND PERIODIC ORBITS

Here we revisit basic facts about pseudo-holomorphic maps proved in [19] for the contact case. Consider a stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega)$ defined on a manifold N and a compact smooth submanifold with boundary $K \subset N$ with $\dim K = \dim N$. We fix $L > 0$, $J \in \mathcal{J}(\mathcal{H})$, and consider $\mathcal{H}_n^\pm = (\xi_n^\pm, X_n^\pm, \omega_n^\pm) \rightarrow \mathcal{H}$ in C_{loc}^∞ , $J_n^\pm \in \mathcal{J}(\mathcal{H}_n^\pm)$ and $\tilde{J}_n \in \mathcal{J}_L(J_n^-, J_n^+)$ such that $J_n^+, J_n^-, \tilde{J}_n \rightarrow J$ in C_{loc}^∞ , as $n \rightarrow \infty$. It is not hard to construct symplectic forms Ω_n on $[-L, L] \times K$ compatible with J which agree with constant positive multiples of ω_n^\pm on $T(\pm L \times K)$ and which converge, as $n \rightarrow \infty$, to a fixed symplectic form Ω compatible with J on $[-L, L] \times K$. Below we use Ω_n to define the energy of \tilde{J}_n -holomorphic maps. The energy

of a J -holomorphic map $F : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times K$ can be estimated from above and below by the modified energy

$$\begin{aligned} & \sup_{\phi \in \Lambda} \int_{F^{-1}((\mathbb{R} \setminus [-L, L]) \times K)} F^*(d\phi \wedge \lambda) \\ & + \int_{F^{-1}((\mathbb{R} \setminus [-L, L]) \times K)} F^*\omega + \int_{F^{-1}([-L, L] \times K)} F^*\Omega. \end{aligned}$$

Consider the \mathbb{R} -invariant Riemannian metric $g_0 = da \otimes da + \lambda \otimes \lambda + \omega$ on $\mathbb{R} \times N$. Domains in \mathbb{C} or $\mathbb{R} \times S^1$ are equipped with their standard Euclidean metric. Norms of maps are taken with respect to these metrics.

The point of the following lemmas is that X may be very degenerate, so the results from [3] are not available if one wants to analyze sequences of \tilde{J}_n -holomorphic maps with bounded energy. In any case, we note that the following arguments are contained in [19].

Lemma A.1. *Let $F_n = (a_n, f_n) : (\mathbb{C}, i) \rightarrow \mathbb{R} \times K$ be smooth \tilde{J}_n -holomorphic maps satisfying $E(F_n) \leq C, \forall n$. Suppose $\exists \{z_n\} \subset \mathbb{C}$ such that $|dF_n(z_n)| \rightarrow \infty$. Then one finds sequences $\{z'_j\} \subset \mathbb{C}$, $\{\delta_j\}, \{R_j\}, \{d_j\} \subset \mathbb{R}$, and a subsequence F_{n_j} such that $|z_{n_j} - z'_j| \rightarrow 0$, $\delta_j \rightarrow 0^+$, $R_j \rightarrow +\infty$, $\delta_j R_j \rightarrow +\infty$ and maps*

$$\tilde{u}_j : B_{\delta_j R_j}(0) \rightarrow \mathbb{R} \times N, \quad \tilde{u}_j(z) = \tau_{d_j} \circ F_{n_j}(z'_j + z/R_j)$$

converge in C_{loc}^∞ to a J -holomorphic map $\tilde{u} : \mathbb{C} \rightarrow \mathbb{R} \times N$ satisfying $0 < E(\tilde{u}) \leq C$ and $\sup_{z \in \mathbb{C}} |d\tilde{u}(z)| < \infty$.

Proof. By Hofer's Lemma (Lemma 5.12 from [3]), there exists $z'_n \in \mathbb{C}$, $\delta_n \rightarrow 0^+$ such that $|z'_n - z_n| \rightarrow 0$ and if we set $R_n = |dF_n(z'_n)|$ then $R_n \rightarrow \infty$, $\delta_n R_n \rightarrow \infty$ and $|dF_n| \leq 2R_n$ on $B_{\delta_n}(z'_n)$. We denote $d_n = -a_n(z'_n)$ and define $\tilde{u}_n = \tau_{d_n} \circ F_n(z'_n + z/R_n)$ on $B_{\delta_n R_n}(0) \subset \mathbb{C}$. Thus $\tilde{u}_n(0) \in 0 \times N$ and $|d\tilde{u}_n| \leq 2$ on $B_{\delta_n R_n}(0)$, for all n . Clearly \tilde{u}_n is $(\tau_{-d_n}^* \tilde{J}_n)$ -holomorphic and $\tau_{-d_n}^* \tilde{J}_n \rightarrow J$ in C_{loc}^∞ . Elliptic estimates provide C_{loc}^∞ -bounds for the sequence $\{\tilde{u}_n\}$ so, up to selection of a subsequence, we may assume $\tilde{u}_n \rightarrow \tilde{u}$ in C_{loc}^∞ , where \tilde{u} is J -holomorphic. Then $|d\tilde{u}(0)| = 1$ and $|d\tilde{u}(z)| \leq 2 \forall z$ since the same holds for \tilde{u}_n . One easily checks $E(\tilde{u}) \leq C$. \square

Remark A.2. *The proof of Lemma A.1 shows that one can replace the domain (\mathbb{C}, i) of the maps F_n by (\mathbb{D}, i) and assume $z_n \rightarrow 0$, or by $([0, +\infty) \times \mathbb{R}/\mathbb{Z}, i)$ and assume $z_n = (s_n, t_n)$ satisfies $s_n \rightarrow +\infty$. The conclusion is exactly the same in both cases.*

Lemma A.3. *Let $F = (a, f) : \mathbb{C} \rightarrow \mathbb{R} \times K$ be a non-constant J -holomorphic map satisfying $\int_{\mathbb{C}} f^*\omega = 0$. Then there exists a (not necessarily periodic) trajectory x of X and an entire function $H : \mathbb{C} \rightarrow \mathbb{C}$ such that $F = Z^x \circ H$ where the J -holomorphic immersion $Z^x : \mathbb{C} \rightarrow \mathbb{R} \times N$ is defined by $Z^x(s + it) = (s, x(t))$. If, in addition, $|dF|$ is bounded then $H(z) = \alpha z + \beta$ with $\alpha \neq 0$.*

Proof. The identity $\bar{\partial}_J(F) = 0$ tells us $\int_{\mathbb{C}} f^*\omega = 0 \Rightarrow f^*\omega \equiv 0$, df takes values on $\mathbb{R}X \circ f$ and $da(z) = 0 \Leftrightarrow df(z) = 0 \Leftrightarrow dF(z) = 0$. Fix $z_1, z_0 \in \mathbb{C}$ and any smooth curve $z(t) : (-\epsilon, 1 + \epsilon) \rightarrow \mathbb{C}$ satisfying $z(0) = z_0$ and $z(1) = z_1$. Let $x : \mathbb{R} \rightarrow N$ be the trajectory of X satisfying $x(0) = f(z_0)$. There is a unique function $g(t)$ defined by $df(z(t)) \cdot z'(t) = g(t)X \circ f \circ z(t)$ since df takes values on $\mathbb{R}X \circ f$. Then $Y(t, p) = g(t)X(p)$ defines a time-dependent vector field on $(-\epsilon, 1 + \epsilon) \times N$. Consider $h(t) := \int_0^t g(\tau) d\tau$. Then $f \circ z$ and $x \circ h$ solve $\beta' = Y(t, \beta)$ with the same initial condition, and hence are equal. Since z_1 was arbitrary it follows that $F(\mathbb{C}) \subset Z^x(\mathbb{C})$.

Assume x is not periodic. Then there exists a unique function $H : \mathbb{C} \rightarrow \mathbb{C}$ satisfying $F = Z^x \circ H$ because Z^x is 1-1. Since Z^x a J -holomorphic immersion we conclude, using the similarity principle, that H is holomorphic, see Lemma 2.4.3 from [26]. When x is periodic, the map Z^x descends to a map \underline{Z}^x defined on $\mathbb{R} \times \mathbb{R}/T\mathbb{Z}$, where $T > 0$ is the minimal period of x . As before there exists a unique holomorphic function $\underline{H} : \mathbb{C} \rightarrow \mathbb{R} \times \mathbb{R}/T\mathbb{Z}$ satisfying $F = \underline{Z}^x \circ \underline{H}$ since \underline{Z}^x is 1-1. Clearly \underline{H} can be lifted to a holomorphic map $H : \mathbb{C} \rightarrow \mathbb{C}$ satisfying $F = Z^x \circ H$.

If $w \in \mathbb{C}$ and $\zeta \in T_w\mathbb{C}$ then there is an estimate $|\zeta| \leq k|dZ^x(w) \cdot \zeta|$, the constant k being independent of w, ζ . This follows very easily from the particular form of the function Z^x and the nature of the metric g_0 . Thus $|dH|$ is bounded if so is $|dF|$, and the conclusion follows from Liouville's Theorem. \square

Lemma A.4. *If $F : \mathbb{C} \rightarrow \mathbb{R} \times K$ satisfies $\bar{\partial}_J(F) = 0$, $\sup_{z \in \mathbb{C}} |dF(z)| < \infty$ and $0 < E(F) < \infty$ then $E_\omega(F) > 0$.*

Proof. If $E_\omega(F) = 0$ then, by Lemma A.3, there exists a trajectory x such that $F(z) = Z^x(\alpha z + \beta)$ for some $\alpha \neq 0$. This implies $E(F) = \infty$. \square

The next lemma is an important characterization of non-constant finite-energy cylinders in cylindrical cobordisms.

Lemma A.5. *Let $F = (a, f) : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times N$ be a non-constant finite-energy J -holomorphic map satisfying $E_\omega(F) = 0$. Then there exists $\gamma = (x, T) \in \mathcal{P}(\mathcal{H})$, constants $a_0, b_0 \in \mathbb{R}$ and a sign $\epsilon = \pm 1$ such that $F(s, t) = (\epsilon Ts + a_0, x(\epsilon Tt + b_0))$.*

Proof. We can think of $F(s + it)$ defined on \mathbb{C} and 1-periodic in t . We claim that $|dF|$ is bounded. If not, let z_n satisfy $|dF(z_n)| \rightarrow \infty$ and consider $F_n(z) := \tau_{-c_n} \circ F(z + z_n)$ with $c_n = a(z_n)$. By Lemma A.1 applied to F_n and Lemma A.4 we can assume, up to the choice of a subsequence, that there exists $r_n \rightarrow 0^+$ and $z'_n \in \mathbb{C}$ satisfying $|z'_n - z_n| \rightarrow 0$ and

$$\liminf_{n \rightarrow \infty} \int_{B_{r_n}(z'_n)} f^* \omega > 0$$

contradicting our hypotheses. By Lemma A.3 we find a trajectory x , $\epsilon = \pm 1$ and constants $T > 0$, $\alpha = \epsilon T + ib$, $\beta = a_0 + ib_0$ such that $F(s + it) = Z^x(\alpha z + \beta) = (\epsilon Ts - bt + a_0, x(bs + \epsilon Tt + b_0))$. Since F is 1-periodic on t we have $b = 0$ and x is T -periodic. \square

We wish to show that, as in the contact case [19], for finite-energy curves with image in K it is possible to distinguish between positive/negative punctures, and non-removable punctures give periodic orbits for X . In order to do so, we assume that ω has a primitive α on a neighborhood of K such that $\inf_K i_X \alpha > 0$. Note that α is a contact form near K . In the language of [20] this means that X is Reeb-like near K since it is a positive multiple of the Reeb vector field of α .

Following [19], if $F = (a, f) : [0, +\infty) \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times K$ is a finite energy J -holomorphic map then the limit

$$m := \lim_{s \rightarrow \infty} \int_{\mathbb{R}/\mathbb{Z}} f^* \alpha$$

exists since $E_\omega(F) < \infty$. Under these assumptions the following important result, due to Hofer [19] in the contact case, holds.

Lemma A.6. *Assume $m \neq 0$ and let $\epsilon = \pm 1$ be its sign. Then $a(s, t) \rightarrow \epsilon\infty$ as $s \rightarrow \infty$, and $\forall s_n \rightarrow \infty$ there exist $n_j \rightarrow \infty$, a periodic orbit $\gamma = (x, T) \in \mathcal{P}(\mathcal{H})$ and $c \in \mathbb{R}$ such that $f(s_{n_j}, t) \rightarrow x(\epsilon Tt + c)$ in C^∞ as $j \rightarrow \infty$.*

We give a proof here since the statement above can not be found in the literature. Note the difference with the results from [19]: X is not the Reeb vector of α near K , and ξ is not a contact structure (it might even be integrable).

Proof. First we show $|dF|$ is bounded. If not let (ρ_n, t_n) satisfy $|dF(\rho_n, t_n)| \rightarrow \infty$ and $|\rho_n| \rightarrow \infty$. Define $F_n(s, t) := F(s + \rho_n, t)$ and write $F_n = (a_n, f_n)$. It follows from $E(F) < \infty$ that $\int_C f_n^* \omega \rightarrow 0$ for every compact $C \subset \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. A combined application of lemmas A.1 and A.4 shows that $|dF_n|$ is bounded over compact sets, contradicting $|dF_n(0, t_n)| \rightarrow \infty$.

Suppose $m > 0$. Define $F_n(s, t) := \tau_{c_n} \circ F(s + s_n, t)$ with $c_n = -a(s_n, 0)$. Thus, by the above, F_n is C_{loc}^1 -bounded and elliptic estimates tell us it is C_{loc}^∞ -bounded. We find $n_j \rightarrow \infty$ and a smooth J -holomorphic map $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R} \times N$ such that $F_{n_j} \rightarrow u$ in C_{loc}^∞ as $j \rightarrow \infty$. Clearly $E(u) \leq E(F)$, $E_\omega(u) = 0$ and $\text{image}(u) \subset \mathbb{R} \times K$. Moreover u is non-constant since $\int_{0 \times \mathbb{R}/\mathbb{Z}} u^* \alpha = m > 0$. By Lemma A.5 we have $u(s, t) = (Ts + a_0, x(Tt + b_0))$, for some $(x, T) \in \mathcal{P}(\mathcal{H})$ contained in K . Here we used the fact that $\inf_K i_X \alpha > 0$ to conclude that sign in Lemma A.5 is $+1$. Clearly $m = T$.

It remains only to show that $a(s, t) \rightarrow +\infty$ as $s \rightarrow \infty$. Consider the mean $\bar{a}(s) := \int_0^1 a(s, t) dt$ and

$$\sigma := \min\{T > 0 \mid \exists \gamma \in \mathcal{P}(\mathcal{H}) \text{ contained in } K \text{ with period } T\} > 0.$$

We claim $\liminf_{s \rightarrow \infty} \bar{a}'(s) \geq \sigma$. If not let $s_n \rightarrow \infty$ satisfy $\sup_n \bar{a}'(s_n) \leq \sigma - \epsilon$. Arguing as above we can assume, up to the choice of a subsequence, that $f(s_n, t) \rightarrow x(Tt + c)$ in C^∞ , for some $\gamma = (x, T) \in \mathcal{P}(\mathcal{H})$ contained in K , and $c \in \mathbb{R}$. Let λ be the 1-form defined by (2.1). Then

$$\begin{aligned} \sigma - \epsilon &\geq \limsup_{n \rightarrow \infty} \bar{a}'(s_n) = \limsup_{n \rightarrow \infty} \int_0^1 a_s(s_n, t) dt \\ &= \limsup_{n \rightarrow \infty} \int_0^1 \lambda(f(s_n, t)) \cdot f_t(s_n, t) dt = T \geq \sigma. \end{aligned}$$

This contradiction proves our claim. Thus $\bar{a}(s) \rightarrow +\infty$ as $s \rightarrow +\infty$. We conclude the case $m > 0$ since $|a(s, t) - \bar{a}(s)|$ is uniformly bounded by $\sup |a_t| < \infty$. The case $m < 0$ is treated similarly. \square

The following statement, left with no proof, follows easily from the assumption that ω has a primitive on K .

Lemma A.7. *If S is a closed Riemann surface, $M \subset S$ is finite, and $F = (a, f) : S \setminus M \rightarrow \mathbb{R} \times K$ is a non-constant finite-energy J -holomorphic then $M \neq \emptyset$ and at least one point of M is a non-removable positive puncture.*

REFERENCES

- [1] M. Abreu and L. Macarini, *Contact homology of good toric contact manifolds*. *Compositio Math.* **148** (2012), no. 1, 304–334.
- [2] F. Bourgeois, *A survey of contact homology*, in *New perspectives and challenges in symplectic field theory*. 45–71, CRM Proc. Lecture Notes, 49, Amer. Math. Soc., Providence, RI, 2009.
- [3] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, *Compactness results in Symplectic Field Theory*. *Geometry and Topology* **7** (2004), 799–888.

- [4] F. Bourgeois and K. Monkhke, *Coherent orientations in Symplectic Field Theory*. Math. Z. **248** (2004) no. 1, 123–146.
- [5] F. Bourgeois and A. Oancea, *An exact sequence for contact and symplectic homology*. Invent. Math. **175** (2009), 611–680.
- [6] F. Bourgeois and O. van Koert, Work in progress.
- [7] K. Cieliebak and J. Latschev, *The role of string topology in symplectic field theory* in *New perspectives and challenges in symplectic field theory*, 113146, CRM Proc. Lecture Notes, 49, Amer. Math. Soc., Providence, RI, 2009.
- [8] V. Colin and K. Honda, *Reeb vector fields and open book decompositions*. ArXiv:0809.5088.
- [9] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*. Geom. Funct. Anal. (2000), Special Volume, Part II, 560–673.
- [10] Y. Eliashberg and L. Traynor, *Symplectic geometry and topology*. IAS/Park City Mathematics Series, Vol. **7**, A.M.S. 1999.
- [11] A. Floer, *Symplectic fixed points and holomorphic spheres*. Comm. Math. Phys. **120** (1989), no. 4, 575–611.
- [12] A. Floer, *Morse theory for Lagrangian intersections*. J. Differential Geom. **28** (1988), no. 3, 513–547.
- [13] A. Floer, *The unregularized gradient flow of the symplectic action*. Comm. Pure Appl. Math. **41** (1988), no. 6, 775–813.
- [14] H. Geiges, *An introduction to contact topology*. Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008.
- [15] V. Ginzburg *The Conley conjecture*. Ann. of Math. (2) **172** (2010), no. 2, 1127–1180.
- [16] V. Ginzburg and B. Gürel, *Local Floer homology and the action gap*. J. Symplectic Geom. **8** (2010), no. 3, 323–357.
- [17] V. Ginzburg and E. Kerman, *Homological resonances for Hamiltonian diffeomorphisms and Reeb flows*. Int. Math. Res. Not. IMRN 2010, no. 1, 53–68.
- [18] D. Gromoll and W. Meyer, *Periodic geodesics on compact riemannian manifolds*. J. Differential Geometry **3** 1969 493–510.
- [19] H. Hofer, *Pseudoholomorphic curves in symplectisations with application to the Weinstein conjecture in dimension three*. Invent. Math. **114** (1993), 515–563.
- [20] H. Hofer, *Holomorphic curves and dynamics in dimension three*, in [10], 35–101.
- [21] H. Hofer, K. Wysocki and E. Zehnder, *Properties of pseudoholomorphic curves in symplectisations I: Asymptotics*. Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 337–379.
- [22] H. Hofer, K. Wysocki and E. Zehnder, *A general Fredholm theory. I. A splicing-based differential geometry*. J. Eur. Math. Soc. **9** (2007), 841–876.
- [23] H. Hofer, K. Wysocki and E. Zehnder, *A general Fredholm theory. II. Implicit function theorems*. Geom. Funct. Anal. **19** (2009), 206–293.
- [24] H. Hofer, K. Wysocki and E. Zehnder, *A general Fredholm theory. III. Fredholm functors and polyfolds*. Geom. Topol. **13** (2009), 2279–2387.
- [25] L. Macarini and G. Paternain *Equivariant symplectic homology of Anosov contact structures*. ArXiv:1111.5579.
- [26] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*. Amer. Math. Soc. Colloq. Publ. **52** (2004).
- [27] M. McLean, *The growth rate of symplectic homology and affine varieties*. ArXiv:1011.2542.
- [28] M. McLean, *Local Floer homology and infinitely many simple Reeb orbits*. ArXiv:1202.0528.
- [29] D. Salamon, *Lectures on Floer homology*, in [10], 145–229.
- [30] P. Seidel, *A biased view of symplectic cohomology*. Current developments in mathematics, 2006, 211253, Int. Press, Somerville, MA, 2008.
- [31] M. Vigué-Poirrier and D. Sullivan, *The homology theory of the closed geodesic problem*. J. Differential Geometry **11** (1976), no. 4, 633–644.
- [32] C. Viterbo, *Equivariant Morse theory for starshaped Hamiltonian systems*. Trans. Amer. Math. Soc. **311** (1989), no. 2, 621–655.
- [33] C. Wendl, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*. Comment. Math. Helv. **85** (2010), no. 2, 347–407.
- [34] M. Yau, *Cylindrical contact homology of subcritical Stein-fillable contact manifolds*. Geom. Topol. **8** (2004), 1243–1280.

UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, INSTITUTO DE MATEMÁTICA, CIDADE UNIVERSITÁRIA, CEP
21941-909 - RIO DE JANEIRO - BRAZIL

E-mail address: `umberto@labma.ufrj.br`

UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, INSTITUTO DE MATEMÁTICA, CIDADE UNIVERSITÁRIA, CEP
21941-909 - RIO DE JANEIRO - BRAZIL

E-mail address: `leonardo@impa.br`